Geodesic equation — curvature

- Analogous to gauge field strength

$$D_\mu\psi = \left(\partial_\mu + \frac{ie}{c\hbar}A_\mu\right)\psi \;\to\; [D_\mu, D_\nu] = (\partial_\mu A_\nu - \partial_\nu A_\mu)\frac{ie}{\hbar c}$$
$$= \frac{ie}{\hbar c} F_{\mu\nu}.$$

✱ Covariant derivative over vector

$$D_\mu A^\rho = \partial_\mu A^\rho + \Gamma_\mu{}^\rho{}_\lambda A^\lambda$$

$$D_\nu(D_\mu A^\rho) = \partial_\nu(D_\mu A^\rho) - \Gamma^\lambda_{\nu\mu} D_\lambda A^\rho + \Gamma^\rho_{\nu\lambda} D_\mu A^\lambda$$
$$= \partial_\nu(\partial_\mu A^\rho + \Gamma^\rho_{\mu\lambda} A^\lambda) - \Gamma^\lambda_{\nu\mu}(\partial_\lambda A^\rho + \Gamma^\rho_{\lambda\sigma} A^\sigma) + \Gamma^\rho_{\nu\lambda}(\partial_\mu A^\lambda + \Gamma^\lambda_{\mu\sigma} A^\sigma)$$
$$= \partial_\nu\partial_\mu A^\rho + \Gamma^\rho_{\mu\lambda}\partial_\nu A^\lambda - \Gamma^\lambda_{\nu\mu}\partial_\lambda A^\rho + \Gamma^\rho_{\nu\lambda}\partial_\mu A^\lambda$$
$$+ \partial_\nu\Gamma^\rho_{\mu\lambda} A^\lambda + (-\Gamma^\lambda_{\nu\mu}\Gamma^\rho_{\lambda\sigma} + \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}) A^\sigma$$

exchange $\mu \leftrightarrow \nu$

$$D_\mu(D_\nu A^\rho) = \partial_\mu\partial_\nu A^\rho + \Gamma^\rho_{\nu\lambda}\partial_\mu A^\lambda - \Gamma^\lambda_{\nu\mu}\partial_\lambda A^\rho + \Gamma^\rho_{\mu\lambda}\partial_\nu A^\lambda$$
$$+ \partial_\mu\Gamma^\rho_{\nu\lambda} A^\lambda + (-\Gamma^\lambda_{\nu\mu}\Gamma^\rho_{\lambda\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma}) A^\sigma$$

$$[D_\mu, D_\nu]A^\rho = \left[\partial_\mu\Gamma^\rho_{\nu\sigma} - \partial_\nu\Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}\right] A^\sigma = R^\rho{}_{\sigma\mu\nu} A^\sigma$$

where we define

$$\boxed{R^\rho{}_{\sigma\mu\nu} = \partial_\mu\Gamma^\rho_{\sigma\nu} - \partial_\nu\Gamma^\rho_{\sigma\mu} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\sigma\nu} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\sigma\mu}}$$

independent of "A" itself, only related to "metric"!

$\Rightarrow \quad R^\rho{}_{\sigma\mu\nu} = -R^\rho{}_{\sigma\nu\mu}$ anti-symmetric under the exchange $\mu \leftrightarrow \nu$.

(2) $$D_\mu A_\rho = \partial_\mu A_\rho - \Gamma^\lambda_{\mu\rho} A_\lambda$$

$$D_\nu(D_\mu A_\rho) = \partial_\nu(D_\mu A_\rho) - \Gamma^\lambda_{\nu\mu} D_\lambda A_\rho - \Gamma^\lambda_{\nu\rho} D_\mu A_\lambda$$

$$= \partial_\nu[\partial_\mu A_\rho - \Gamma^\lambda_{\mu\rho}A_\lambda] - \Gamma^\lambda_{\nu\mu}[\partial_\lambda A_\rho - \Gamma^\sigma_{\lambda\rho}A_\sigma] - \Gamma^\lambda_{\nu\rho}[\partial_\mu A_\lambda - \Gamma^\sigma_{\mu\lambda}A_\sigma]$$

$$= \partial_\nu\cancel{\partial_\mu} A_\rho - \cancel{\Gamma^\lambda_{\mu\rho}}\partial_\nu A_\lambda - \Gamma^\lambda_{\nu\mu}\cancel{\partial_\lambda} A_\rho - \Gamma^\lambda_{\nu\rho}\cancel{\partial_\mu}A_\lambda$$

$$- \partial_\nu \Gamma^\lambda_{\mu\rho} A_\lambda + (\cancel{\Gamma^\lambda_{\nu\mu}}\Gamma^\sigma_{\lambda\rho} + \Gamma^\lambda_{\nu\rho}\Gamma^\sigma_{\mu\lambda})A_\sigma$$

$$D_\mu D_\nu A_\rho = \partial_\mu\cancel{\partial_\nu}A_\rho - \cancel{\Gamma^\lambda_{\nu\rho}}\partial_\mu A_\lambda - \cancel{\Gamma^\lambda_{\mu\nu}}\partial_\lambda A_\rho - \cancel{\Gamma^\lambda_{\mu\rho}}\partial_\nu A_\lambda$$

$$- \partial_\mu\Gamma^\lambda_{\nu\rho}A_\lambda + (\cancel{\Gamma^\lambda_{\mu\nu}}\Gamma^\sigma_{\lambda\rho} + \Gamma^\lambda_{\mu\rho}\Gamma^\sigma_{\nu\lambda})A_\sigma$$

$$[D_\mu, D_\nu]A_\rho = [-\partial_\mu\Gamma^\sigma_{\nu\rho} + \partial_\nu\Gamma^\sigma_{\mu\rho} + \Gamma^\lambda_{\mu\rho}\Gamma^\sigma_{\nu\lambda} - \Gamma^\lambda_{\nu\rho}\Gamma^\sigma_{\mu\lambda}]A_\sigma$$

$$\boxed{[D_\mu, D_\nu]A_\rho = -R^\sigma{}_{\rho\mu\nu}A_\sigma}$$

☆ Covariant derivatives on a tensor

$$D_\mu D_\nu T^{\lambda\rho} = \partial_\mu(D_\nu T^{\lambda\rho}) - \Gamma^\sigma_{\mu\nu}\cancel{D_\sigma}T^{\lambda\rho} + \Gamma^\lambda_{\mu\sigma}D_\nu T^{\sigma\rho} + \Gamma_\mu{}^\rho{}_\sigma D_\nu T^{\lambda\sigma}$$

$$D_\nu D_\mu T^{\lambda\rho} = \partial_\nu(D_\mu T^{\lambda\rho}) - \Gamma^\sigma_{\nu\mu}\cancel{D_\sigma}T^{\lambda\rho} + \Gamma^\lambda_{\nu\sigma}D_\mu T^{\sigma\rho} + \Gamma^\rho_{\nu\sigma}D_\mu T^{\lambda\sigma}$$

$$[D_\mu,D_\nu]T^{\lambda\rho} = \partial_\mu[\partial_\nu T^{\lambda\rho} + \Gamma^\lambda_{\nu\sigma}T^{\sigma\rho} + \Gamma^\rho_{\nu\sigma}T^{\lambda\sigma}]$$

$$+ \Gamma^\lambda_{\mu\sigma}[\partial_\nu T^{\sigma\rho} + \Gamma^\sigma_{\nu\sigma'}T^{\sigma'\rho} + \Gamma^\rho_{\nu\sigma'}T^{\sigma\sigma'}]$$

$$+ \Gamma^\rho_{\mu\sigma}[\partial_\nu T^{\lambda\sigma} + \Gamma^\lambda_{\nu\sigma'}T^{\sigma'\sigma} + \Gamma^\sigma_{\nu\sigma'}T^{\lambda\sigma'}] - (\mu\leftrightarrow\nu)$$

$$= \partial_\mu\partial_\nu T^{\lambda\rho} + \partial_\mu\Gamma^\lambda_{\nu\sigma}T^{\sigma\rho} + \Gamma^\lambda_{\nu\sigma}\partial_\mu T^{\sigma\rho} + \partial_\mu\Gamma^\rho_{\nu\sigma}T^{\lambda\sigma} + \Gamma^\rho_{\nu\sigma}\partial_\mu T^{\lambda\sigma}$$

$$\Gamma^\lambda_{\mu\sigma}\partial_\nu T^{\sigma\rho} + \qquad + \Gamma^\rho_{\mu\sigma}\partial_\nu T^{\lambda\sigma}$$

$$+ \underline{\Gamma^\lambda_{\mu\sigma}\Gamma^\rho_{\nu\sigma'}T^{\sigma\sigma'} + \Gamma^\rho_{\mu\sigma}\Gamma^\lambda_{\nu\sigma'}T^{\sigma'\sigma}} + \Gamma^\lambda_{\mu\sigma}\Gamma^\sigma_{\nu\sigma'}T^{\sigma'\rho} + \Gamma^\rho_{\mu\sigma}\Gamma^\sigma_{\nu\sigma'}T^{\lambda\sigma'} - (\mu\leftrightarrow\nu)$$

$$[D_\mu, D_\nu] T^{\lambda\rho} = \left(\partial_\mu \Gamma^\lambda_{\nu\sigma} - \partial_\nu \Gamma^\lambda_{\mu\sigma} + \Gamma^\lambda_{\mu\sigma'} \Gamma^{\sigma'}_{\nu\sigma} - \Gamma^\lambda_{\nu\sigma'} \Gamma^{\sigma'}_{\nu\sigma}\right) T^{\sigma\rho}$$

$$+ \left(\partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\sigma'} \Gamma^{\sigma'}_{\nu\sigma} - \Gamma^\rho_{\nu\sigma'} \Gamma^{\sigma'}_{\mu\sigma}\right) T^{\lambda\sigma}$$

$$\boxed{[D_\mu, D_\nu] T^{\lambda\rho} = R^\lambda{}_{\sigma\mu\nu} T^{\sigma\rho} + R^\rho{}_{\sigma\mu\nu} T^{\lambda\sigma}}$$

(*) $$\boxed{R^\rho{}_{\lambda\mu\nu} + R^\rho{}_{\mu\nu\lambda} + R^\rho{}_{\nu\lambda\mu} = 0}$$

Proof: $R^\rho{}_{\lambda\mu\nu} = \partial_\mu \Gamma^\rho_{\lambda\nu} - \partial_\nu \Gamma^\rho_{\lambda\mu} + \Gamma^\rho_{\mu\sigma}\Gamma^\sigma_{\lambda\nu} + \Gamma^\rho_{\nu\sigma}\Gamma^\sigma_{\lambda\mu}$

$R^\rho{}_{\mu\nu\lambda} = \partial_\nu \Gamma^\rho_{\mu\lambda} - \partial_\lambda \Gamma^\rho_{\mu\nu} + \Gamma^\rho_{\nu\sigma}\Gamma^\sigma_{\mu\lambda} + \Gamma^\rho_{\lambda\sigma}\Gamma^\sigma_{\mu\nu}$

$R^\rho{}_{\nu\lambda\mu} = \partial_\lambda \Gamma^\rho_{\nu\mu} - \partial_\mu \Gamma^\rho_{\nu\lambda} + \Gamma^\rho_{\lambda\sigma}\Gamma^\sigma_{\nu\mu} + \Gamma^\rho_{\mu\sigma}\Gamma^\sigma_{\nu\lambda}$

Add together $= 0$

Example: consider $d\tau^2 = \frac{1}{t^2}(dt^2 - dx^2)$

$g_{tt} = 1/t^2, \quad g_{xx} = -1/t^2, \quad g^{tt} = t^2, \quad g^{xx} = -t^2$

$$\Gamma^\sigma_{\mu\nu} = \frac{1}{2} g^{\sigma\lambda}(\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu})$$

$$\Gamma^t_{tt} = \frac{1}{2} g^{t\lambda}(\partial_t g_{t\lambda} + \partial_t g_{t\lambda} - \partial_\lambda g_{tt}) = \frac{1}{2} g^{tt}\partial_t g_{tt} = \frac{t^2}{2}\partial_t t^{-2} = -1/t$$

$$\Gamma^t_{xx} = \frac{1}{2} g^{t\lambda}(\partial_x g_{x\lambda} + \partial_x g_{x\lambda} - \partial_\lambda g_{xx}) = -\frac{1}{2} g^{tt}\partial_t g_{xx} = -\frac{1}{2}t^2 \partial_t(-1/t^2) = -1/t$$

$$\Gamma^x_{tx} = \Gamma^x_{xt} = \frac{1}{2} g^{x\lambda}(\partial_x g_{t\lambda} + \partial_t g_{x\lambda} - \partial_\lambda g_{xt}) = \frac{1}{2}[g^{xx}\partial_t g_{xx}] = -\frac{t^2}{2}\partial_t(-1/t^2) = -1/t$$

Geodesic equation

$$\frac{d^2x^\ell}{ds^2} + \Gamma^\ell_{kj}\frac{dx^k}{ds}\frac{dx^j}{ds} = 0$$

$$\frac{d^2t}{ds^2} + \Gamma^t_{tt}\left(\frac{dt}{ds}\right)^2 + \Gamma^t_{xx}\left(\frac{dx}{ds}\right)^2 = 0 \Rightarrow \frac{d^2t}{ds^2} - 1/t\left[\left(\frac{dt}{ds}\right)^2 + \left(\frac{dx}{ds}\right)^2\right] = 0$$

$$\frac{d^2x}{ds^2} + 2\Gamma^x_{xt}\frac{dx}{ds}\frac{dt}{ds} = 0 \Rightarrow \frac{d^2x}{ds^2} - \frac{2}{t}\frac{dx}{ds}\frac{dt}{ds} = 0.$$

$$R^t_{xtx} = \partial_t \Gamma^t_{xx} - \partial_x \Gamma^t_{xt} + \Gamma^t_{t\lambda}\Gamma^\lambda_{xx} - \Gamma^t_{x\lambda}\Gamma^\lambda_{xt}$$

$$= \partial_t[-1/t] + \Gamma^t_{tt}\Gamma^t_{xx} - \Gamma^t_{xx}\Gamma^x_{xt} = 1/t^2 + (-1/t)^2 - (-1/t)(-1/t)^2 = 1/t^2$$

$$R_{txtx} = g_{tt} R^t_{xtx} = 1/t^2 \cdot 1/t^2 = 1/t^4.$$

(*) Parallel transport on a sphere

Consider a vector $A^\alpha = (A^\theta, A^\varphi)$. On a sphere $ds^2 = d\theta^2 + \sin^2\theta d\varphi^2$

$$g_{\theta\theta} = 1, \quad g_{\phi\phi} = \sin^2\theta, \Rightarrow g^{\theta\theta} = 1, \quad g^{\phi\phi} = \frac{1}{\sin^2\theta}$$

$$\Gamma^\sigma_{\mu\nu} = \frac{1}{2} g^{\sigma\lambda}(\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu})$$

$$\Gamma^\theta_{\phi\phi} = \frac{1}{2} g^{\theta\theta}(\partial_\phi g_{\phi\theta} + \partial_\phi g_{\phi\theta} - \partial_\theta g_{\phi\phi}) = -\frac{1}{2}\partial_\theta \sin^2\theta = -\sin\theta\cos\theta$$

$$\Gamma^\phi_{\theta\phi} = \Gamma^\phi_{\phi\theta} = \frac{1}{2} g^{\phi\phi}(\partial_\theta g_{\phi\phi} + \partial_\phi g_{\theta\phi} - \partial_\phi g_{\theta\phi}) = \frac{1}{2}\frac{1}{\sin^2\theta}\partial_\theta \sin^2\theta$$

$$= \cot\theta.$$

Consider transport along $x(\tau) \Rightarrow \dfrac{dx^\nu}{d\tau}(\partial_\nu A^\mu + \Gamma^\mu_{\nu\lambda} A^\lambda) = 0$

Along $\theta = \theta_0$, φ is the variable

$$\partial_\varphi A^\mu + \Gamma^\mu_{\varphi\lambda} A^\lambda = 0 \Rightarrow \begin{cases} \partial_\phi A^\theta + \Gamma^\theta_{\phi\phi}(\theta=\theta_0) A^\phi = 0, \\ \partial_\phi A^\varphi + \Gamma^\phi_{\phi\theta}(\theta=\theta_0) A^\theta = 0. \end{cases}$$

$$\Rightarrow \begin{cases} \partial_\phi A^\theta = \frac{1}{2}\sin 2\theta_0 A^\varphi \\ \partial_\phi A^\phi = -\cot\theta_0 A^\theta \end{cases} \Rightarrow \partial^2_\phi A^\phi = -\cos^2\theta_0 A^\phi$$

If imposing the initial condition $A_\theta^{\theta=\theta_0} = 1$, $A^{\phi=0} = 0$, i.e A along A^θ

$$\Rightarrow A^\phi = C\sin(\phi\cos\theta_0)$$

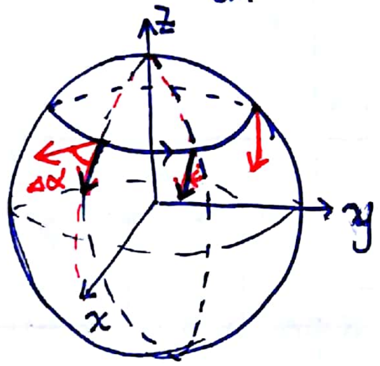
$$A^\theta = -\tan\theta_0\, \partial_\phi A^\varphi = -C\tan\theta_0 \cos(\phi\cos\theta_0)\cos\theta_0 = -C\sin\theta_0 \cos(\phi\cos\theta_0)$$

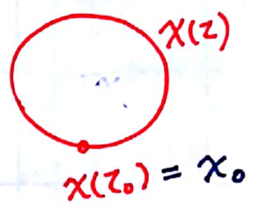
$$\Rightarrow \boxed{\begin{cases} A^\theta = \cos(\phi\cos\theta_0) \\ A^\varphi = -\csc\theta_0 \sin(\phi\cos\theta_0) \end{cases}} \Rightarrow g_{\theta\theta}A^{\theta^2} + g_{\varphi\varphi}A^{\varphi^2} = \cos^2(\phi\cos\theta_0) + \sin^2(\phi\cos\theta_0) = 1$$

However, when the vecter goes round θ_0, $\rightarrow \phi = 2\pi$

$$\begin{cases} A^\theta = -\cos(2\pi\cos\theta_0) \\ A^\varphi = -\csc\theta_0 \underset{\sin}{(2\pi\cos\theta_0)} \end{cases} \longrightarrow \vec{A} = \cos(2\pi\cos\theta_0)\,\hat{e}_\theta - \sin(2\pi\cos\theta_0)\,\hat{e}_\varphi$$

$$\boxed{\Delta\alpha = -2\pi\cos\theta_0}$$





(*) Around an infinitsmall loop, the parallel transport

$$d\xi^\mu = -\Gamma^\mu_{\nu\lambda}\, dx^\nu \xi^\lambda$$

$$\Delta\xi^\mu = \oint d\xi^\mu = \oint d\tau \frac{d\xi^\mu}{d\tau} = -\oint d\tau\, \Gamma^\mu_{\nu\lambda}(x) \frac{dx^\nu}{d\tau} \xi^\lambda(x)$$

$$\Gamma^\mu_{\nu\lambda}(x) = \Gamma^\mu_{\nu\lambda}(x_0) + (x-x_0)^\rho \partial_\rho \Gamma^\mu_{\nu\lambda}(x_0)$$

$$\xi^\mu(x) \simeq \xi^\mu(x_0) - \Gamma^\mu_{\nu\lambda}(x_0)(x-x_0)^\nu \xi^\lambda(x_0) \leftarrow \text{parallel}$$

$$\Delta\xi^\mu = \oint d\tau \; -[\Gamma^\mu_{\nu\lambda}(x_0) + (x-x_0)^\gamma \partial_\gamma \Gamma^\mu_{\nu\lambda}(x_0) + \cdots]$$
$$[\xi^\lambda(x_0) - \Gamma^\lambda_{\rho\sigma}(x-x_0)^\rho \xi^\sigma(x_0)] \frac{dx^\nu}{d\tau}$$

$$= \oint d\tau \; -\Gamma^\mu_{\nu\lambda}(x_0)\xi^\lambda(x_0)\frac{dx^\nu}{d\tau} + (x-x_0)^\rho \xi^\sigma(x_0)\{-\partial_\rho \Gamma^\mu_{\nu\sigma}(x_0) + \Gamma^\mu_{\nu\lambda}(x_0)\Gamma^\lambda_{\rho\sigma}(x_0)\} \frac{dx^\nu}{d\tau}$$

For closed loop $\oint \frac{dx^\nu}{d\tau} = 0$

$$\Delta \xi^\mu = \xi^\sigma(x_0) \{-\partial_\rho \Gamma^\mu_{\nu\sigma}(x_0) + \Gamma^\mu_{\nu\lambda}(x_0) \Gamma^\lambda_{\rho\sigma}(x_0)\} \oint d\tau \, (x-x_0)^\rho \frac{dx^\nu}{d\tau}$$

$$\oint d\tau \, x^\rho \frac{dx^\nu}{d\tau} = - \oint d\tau \, x^\nu \frac{dx^\rho}{d\tau}$$

$$\Rightarrow \Delta \xi^\mu = \frac{1}{2} \xi^\sigma(x_0) [-\partial_\rho \Gamma^\mu_{\nu\sigma}(x_0) + \partial_\nu \Gamma^\mu_{\rho\sigma}(x_0) + \Gamma^\mu_{\nu\lambda}(x_0) \Gamma^\lambda_{\rho\sigma}(x_0) - \Gamma^\mu_{\rho\lambda}(x_0) \Gamma^\lambda_{\nu\sigma}(x_0)]$$

$$\oint d\tau \, x^\rho dx^\nu / d\tau$$

$$= \frac{1}{2} \xi^\sigma(x_0) \, R^\mu{}_{\sigma\nu\rho}(x_0) \underbrace{\oint d\tau \, x^\rho \frac{dx^\nu}{d\tau}}$$ ← area of the loop

Hence, the change of a vector after parallel transport around a loop $\propto$ the curvature tensor and the area of the loop.

$$\boxed{\Delta \xi^\mu = \frac{1}{2} R^\mu{}_{\sigma\nu\rho}(x_0) \, \xi^\sigma A^{\nu\rho} \leftarrow A^{\nu\rho} = \oint dx^\nu x^\rho}$$

① If the curvatures vanish, then $\Delta \xi^\mu = 0. \Rightarrow D_\nu \xi^\mu = 0$

$\Delta \xi^\mu = 0$ means that the parallel transport does not depend on the path. The we define $\xi^\mu(x)$ as the value of $\xi^\mu(x_0)$ being parallelly transported to x. The we have $\frac{d\xi^\mu}{d\tau} = \frac{dx^\nu}{d\tau} \frac{\partial \xi^\mu}{\partial x^\nu}$, we also have $\frac{d\xi^\mu}{d\tau} = - \Gamma^\mu_{\nu\lambda} \frac{dx^\nu}{d\tau} \xi^\lambda$, hence, by equalling these two

$$D_\lambda \xi^\mu = \partial_\nu \xi^\mu + \Gamma^\mu_{\nu\lambda} \xi^\lambda = 0.$$

② Coversly, if $D_\nu \xi^\mu = 0 \Rightarrow [D_\lambda, D_\nu]\xi^\mu = 0 \Rightarrow R^\mu_{\ \sigma\lambda\nu} = 0$

and we can transport ξ^μ along any infinitsmall loop without any change.

⊛ Geodesic equations

we have derived the geodesic equations via variation principles ~~as the minimal length~~ curves of. Now we define the "straightline" on a curved manifold, ~~as its~~ that with tangent vector "parallel" to itself along the curve. in the sense of parallel transport.

Define a vector field $\xi^\mu = dx^\mu/d\tau$, which is the tangent vector.

Then its parallel transport

$$\frac{d\xi^\mu}{d\tau} = -\Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{d\tau}\xi^\lambda, \quad \text{plug in } \xi^\mu = \frac{dx^\mu}{d\tau}$$

$$\Rightarrow \frac{d^2}{d\tau^2}x^\mu + \Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\tau} = 0.$$

Hence, the geodesic curve carries its tangent parallel to itself.

Affine variable: τ is a variable labeling the trajectory, which could be choosen as the proper time. If we chose a new parameter $s(\tau)$

$$\Rightarrow \frac{d}{d\tau}\left(\frac{ds}{d\tau}\frac{dx^\mu}{ds}\right) + \Gamma^\mu_{\nu\lambda}\left(\frac{ds}{d\tau}\frac{dx^\nu}{ds}\right)\left(\frac{ds}{d\tau}\frac{dx^\lambda}{d\tau}\right) = 0$$

$$\frac{d^2x^\mu}{ds^2} + \Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{ds}\frac{dx^\nu}{ds} = -\frac{d^2s/d\tau^2}{\left(\frac{ds}{d\tau}\right)^2}\frac{dx^\mu}{ds}$$

$\Rightarrow$ if $s = a\tau + b$, then the geodesic equation is invariant.

Affine action: the action $S = m\int d\tau \left(g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\right)^{1/2}$. This action only works for the case $g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} > 0$, i.e. time-like trajectories.

To avoid this difficulty, we can use

$$\tilde{S} = \int d\tau\, \mathcal{L} = \frac{1}{2}\int d\tau \left(\frac{1}{\sqrt{F}} g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} + \sqrt{F}\, m^2\right)$$

then it includes all trajectories.

$$\frac{\partial \mathcal{L}}{\partial F} = 0 \Rightarrow -\frac{1}{2F^{3/2}} g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} + \frac{1}{2F^{1/2}} m^2 = 0$$

$$F = \frac{1}{m^2} g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}$$, plug in

$$\rightarrow \hat{S} = m\int d\tau \sqrt{g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}}$$

$\boxed{\circledast \text{ Classic Gravity}}$: Newton $\phi(x) = -GM/x^2$, and $\frac{d^2\vec{x}}{dt^2} = -\nabla\phi(\vec{x})$.

Then we want the geodesics to reproduce the Newton's 2nd law.

If the gravity is weak, we have $g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x)$, and then $g^{\mu\nu}(x) = \eta^{\mu\nu} - h^{\mu\nu}(x)$, such that

$$g^{\mu\nu}(x)\, g_{\nu\lambda}(x) = (\eta^{\mu\nu} - h^{\mu\nu})(\eta_{\nu\lambda} + h_{\nu\lambda}) = \delta^\mu{}_\lambda - h^\mu{}_\lambda + h^\mu{}_\lambda = \delta^\mu{}_\lambda.$$

Since we are considering a static problem, we have

$$\partial_0 g_{\mu\nu} = \partial_0 h_{\mu\nu} = 0.$$

If we look at $\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\tau} = 0$, where '$\tau$' is the propertime. In the non-relativistic ~~limt~~ limit, $\frac{dx^0}{d\tau} = \frac{dt}{d\tau} \gg \frac{dx^i}{d\tau} \Rightarrow$

$$\boxed{\frac{d^2x^\mu}{d\tau^2} - \Gamma^\mu_{00}\frac{dt}{d\tau}\frac{dt}{d\tau} = 0}$$

$$\Gamma^{\mu}_{\nu\lambda} = \tfrac{1}{2} g^{\mu\rho}(\partial_\nu g_{\lambda\rho} + \partial_\lambda g_{\nu\rho} - \partial_\rho g_{\nu\lambda}) \Rightarrow$$

$$\Gamma^{\mu}_{00} = \tfrac{1}{2} g^{\mu\rho}(\partial_0 g_{0\rho} + \partial_0 g_{0\rho} - \partial_\rho g_{00}) = -\tfrac{1}{2} g^{\mu\rho}\partial_\rho g_{00} \simeq -\tfrac{1}{2}\eta^{\mu\rho}\partial_\rho h_{00}$$

$$\Rightarrow \Gamma^{0}_{00} = 0, \quad \Gamma^{i}_{00} \simeq \tfrac{1}{2}\eta^{ii}\partial_i h_{00} = -\tfrac{1}{2}\partial^i h_{00} = \tfrac{1}{2}\nabla_i h_{00}$$

$$\frac{d^2 t}{d\tau^2} = 0, \quad \frac{d^2\vec{x}}{d\tau^2} + \frac{1}{2}\vec{\nabla} h_{00}(x)\left(\frac{dt}{d\tau}\right)^2 = 0$$

$$\frac{d^2 t}{d\tau^2} = 0 \Rightarrow \frac{dt}{d\tau} = \text{const} \Rightarrow \boxed{\frac{d^2\vec{x}}{dt^2} = -\frac{1}{2}\vec{\nabla} h_{00}(x) \quad \Rightarrow h_{00}(x) = 2\phi(x) = -\frac{2GM}{|\vec{x}|c^2}}$$

$$\tau = kt$$

add c here

hence, the metric

$$\boxed{g_{00}(x) = 1 - \frac{2GM}{|\vec{x}|c^2}, \quad g_{ij}(x) = \eta_{ij}}$$

$$d\tau^2 = g_{\mu\nu}\, dx^\mu dx^\nu \Rightarrow \left(\frac{d\tau}{dt}\right)^2 = g_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt} = g_{00} + \eta_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt}$$

$$= \left(1 - \frac{2GM}{|\vec{x}|c^2}\right) - \frac{\dot{x}^2}{c^2}$$

$$\Rightarrow \frac{d\tau}{dt} = 1 - \frac{1}{2}\left(\frac{\dot{x}}{c}\right)^2 - \frac{GM}{|\vec{x}|c^2} = 1 - \frac{1}{2mc^2}(T - V)$$

$$\Rightarrow \tau = \int_{t_1}^{t_2} d\tau = \int dt \frac{d\tau}{dt} = \int dt \left[1 - \frac{1}{2mc^2}(T-V)\right]$$

$$\simeq \text{const} - \frac{1}{2mc^2}\int dt\,(T-V) \simeq -\frac{1}{2mc^2}\int dt\,(T-V) = -\frac{L}{2mc^2}$$

This is the Lagrangian in non-relativistic

and $\delta\tau = 0 \Rightarrow$ the geodesics satisfies the least action principle.

Gravitational red shift

Recap the Doppler shift:

Assume the source is moving at a velocity v to the right away from "O" then consider the time intervals of two events dt_0 and dt_s in the oberserver and the source frames. of light emitting

Since the light emittings are at the source, hence dt_s in the proper time, and is the shortest, i.e

$$dt_s = \sqrt{1-\beta^2}\, dt_0$$

If the time interval between two emitsions Δt_s, then during the correpoding time in the O-frame $\frac{\Delta t_s}{\sqrt{1-\beta^2}}$, the source travels a distance $\frac{v\Delta t_s}{\sqrt{1-\beta^2}}$.

Hence, the time interval received by the observer is

$$\Delta t_0 = \Delta t_s + \frac{1}{c}\frac{v\Delta t_s}{\sqrt{1-\beta^2}} = \frac{1+\beta}{\sqrt{1-\beta^2}}\Delta t_s = \sqrt{\frac{1+\beta}{1-\beta}}\,\Delta t_s$$

$$\Rightarrow \boxed{\frac{\nu_0}{\nu_s} = \frac{\Delta t_s}{\Delta t_0} = \left(\frac{1-\beta}{1+\beta}\right)^{1/2}}$$

hence, in the case of the source is moving away, $\nu_0 < \nu_s$, $\Rightarrow$ red shift.