# Lect 6: Quantum inverse method — Fadeev's method

In last class, we have found

$$\boxed{\left[ S_{j+1,j}\, S_{j+2,j} \cdots S_{N,j}\, S_{1j}\, S_{2j} \cdots S_{j-1,j} \right] A_{\sigma_1 \cdots \sigma_N}(12\cdots N, 12\cdots N) = e^{ik_j L} A_{\sigma_1 \cdots \sigma_N}(12\cdots ; 12\cdots}$$

with $S_{ij} = \dfrac{k_i - k_j + ic\, P_{\sigma_i \sigma_j}}{k_i - k_j + ic}$

In order to solve this problem, we use the following method of algebraic BA. We define an auxilary space A, and $\vec{\tau}$ as a Pauli matrix in such a space. Define $P^{j,A} = \frac{1}{2}(1 + \vec{\sigma}_j \cdot \vec{\tau})$, and the auxillany S-matrix

$$\boxed{S^{jA}[u] = \frac{k_j - cu}{k_j - cu + ic} + \frac{ic}{k_j - cu + ic} P^{j,A}}$$

The monodromy matrix

$T(u) = S^{1A}[u]\, S^{2A}[u] \cdots S^{NA}[u]$, where the matrix product only acts in the auxillany space A.

$$T_{\sigma_1 \sigma_2 \cdots \sigma_N u;\, \sigma_1' \sigma_2' \cdots \sigma_N' v} = S^{1A}_{\sigma_1 u;\, \sigma_1' \omega_1}[u]\; S^{2A}_{\sigma_2 \omega_1;\, \sigma_2' \omega_2}[u] \cdots S^{NA}_{\sigma_N \omega_{N-1};\, \sigma_N' v}[u]$$

Trace over the auxillany space $\Rightarrow$

$$tr_A [T(u_j)] = S_{j+1,j}\, S_{j+2,j} \cdots S_{N,j}\, S_{1,j}\, S_{2j} \cdots S_{j-1,j}$$

where $u_j = k_j / c$.

Proof: $tr_A[T(u_j)] = S^{1A}_{\sigma_1 u_1, \sigma_1' u_2} S^{2A}_{\sigma_2 u_2, \sigma_2' u_3} \cdots S^{NA}_{\sigma_N u_N, \sigma_N' u_1}$

we can cyclic rotation over trace indices in the A-space

$$tr_A[T(u_j)] = tr_A[S^{j+1,A}[u_j] S^{j+2,A} \cdots S^{N,A} S^{1A} \cdots S^{j-1,A} S^{j,A}[u_j]]$$

since $S^{j,A}(u_j) = \dfrac{k_j - cu_j + icP^{j,A}}{k_j - cu_j + ic} = P^{j,A}$

and we can prove an identity that $\boxed{S^{lA}(u_j) P^{jA} = P^{jA} S^{jl}(u_j)}$

Then
$$tr_A[T(u_j)] = tr_A[S^{j+1,A}[u_j] \cdots S^{N,A}[u_j] S^{1A}[u_j] \cdots S^{j-1,A}[u_j] P^{jA}]$$
$$= tr_A[S^{j+1,A}[u_j] \cdots\cdots S^{j-2,A} P^{jA} S^{j-1,j}[u_j]]$$
$$= tr_A[P^{jA}] \quad S^{j+1,j}[u_j] S^{j+2,j}[u_j] \cdots S^{N,j}[u_j] S^{1,j} \cdots S^{j-1,j}[u_j]$$

$P^{j,A} = \frac{1}{2}(1 + \vec{\sigma}_j \cdot \vec{\tau}) \Rightarrow tr_A[P^{j,A}] = 1$

$\Rightarrow \boxed{tr_A[T(u_j)] = S_{j+1,j} \cdots S_{N,j} S_{1,j} \cdots S_{j-1,j}}$, where $u_j = k_j/c$.

check: $S^{lA}(u_j) P^{jA} = P^{jA} S^{jl}(u_j) \longrightarrow$ only need to check $P^{lA} P^{jA} = P^{jA} P^{lj}$

$$P^{lA} P^{jA} = \tfrac{1}{4}(1 + \vec{\sigma}_l \cdot \vec{\tau})(1 + \vec{\sigma}_j \cdot \vec{\tau}) = \tfrac{1}{4}(1 + \vec{\sigma}_l \cdot \vec{\tau} + \vec{\sigma}_j \cdot \vec{\tau} + \sigma_l^\alpha \sigma_j^\beta \tau^\alpha \tau^\beta)$$

$$P^{jA} P^{lj} = \tfrac{1}{4}(1 + \vec{\sigma}_j \cdot \vec{\tau})(1 + \vec{\sigma}_l \cdot \vec{\sigma}_j) = \tfrac{1}{4}(1 + \vec{\sigma}_j \cdot \vec{\tau} + \vec{\sigma}_l \cdot \vec{\sigma}_j + \sigma_j^\alpha \sigma_j^\beta \tau^\alpha \sigma_l^\beta)$$

$$\tau^\alpha \tau^\beta = \delta_{\alpha\beta} + i\epsilon_{\alpha\beta\gamma}\tau^\gamma, \quad \sigma_j^\alpha \sigma_j^\beta = \delta_{\alpha\beta} + i\epsilon_{\alpha\beta\gamma}\sigma_j^\gamma$$

$\Rightarrow$ it's easy to check $\boxed{P^{lA} P^{jA} = P^{jA} P^{lj}}$

Then the Bethe ansatz equation is reduced to

$$\boxed{\left[tr_A T(k_j/c)\right]_{\sigma_1\cdots\sigma_N,\,\sigma_1'\cdots\sigma_N'} A_{\sigma_1'\cdots\sigma_N'}(12\cdots N;12\cdots N) = e^{ik_jL} A_{\sigma_1\cdots\sigma_N}^{(12\cdots N,12\cdots}}$$

we need to diagonalize $tr_A T(k_j/c)$ simultanously for $j=1,2,3,\cdots N$.

Define $b(x) = \dfrac{-x}{-x+i}$ , $c(x) = \dfrac{i}{-x+i}$

and R-matrix in two auxillary spaces $A\otimes B$

$$R^{AB}(u) = c(u) + b(u)P_{AB} = (b(u) + c(u)P_{AB})P_{AB}$$

$$S_{jA}(u) = b(u-k_j/c) + c(u-k_j/c)\,P_{jA}$$

$$S_{jB}(u) = b(u-k_j/c) + c(u-k_j/c)\,P_{jB}$$

We have the fundamental commutation relation:

$$\boxed{R^{AB}(u-v)\left[S^{jA}(u)\otimes S^{jB}(v)\right] = \left[S^{jA}(v)\otimes S^{jB}(u)\right]R^{AB}(u-v)}$$

Proof: $LHS = P_{AB}\left[\dfrac{v-u+iP_{AB}}{v-u+i}\right]\left[\dfrac{u_j-u+iP_{jA}}{u_j-u+i}\right]\left[\dfrac{u_j-v+iP_{jB}}{u_j-v+i}\right]$

$$= \left[\frac{v-u+iP_{AB}}{v-u+i}\right]\left[\frac{u_j-u+iP_{Bj}}{u_j-u+i}\right]P_{AB}\left[\frac{u_j-v+iP_{jB}}{u_j-v+i}\right]$$

$$= \left[\frac{v-u+iP_{AB}}{v-u+i}\right]\left[\frac{u_j-u+iP_{Bj}}{u_j-u+i}\right]\left[\frac{u_j-v+iP_{jA}}{u_j-v+i}\right]P_{AB}$$

compare with RHS $\left[\dfrac{u_j-v+iP_{jA}}{u_j-v+i}\right]\left[\dfrac{u_j-u+iP_{jB}}{u_j-u+i}\right]\left[\dfrac{v-u+iP_{AB}}{v-u+i}\right]P_{AB}$

Check $P_{AB} P_{Bj} P_{jA} = P_{AB} P_{jA} P_{AB} = P_{jA} P_{jB} P_{AB}$ ✓

$$(v-u) P_{Bj} P_{jA} + (u_j - u) P_{AB} P_{jA} + (u_j - v) P_{AB} P_{Bj}$$

$$= (v-u) P_{AB} P_{Bj} + (u_j - u) P_{AB} P_{jA} + (u_j - v) P_{AB} P_{Bj}$$

$$= (u_j - u) P_{AB} (P_{Bj} + P_{Aj})$$

$$(u_j - v) P_{jB} P_{AB} + (u_j - u) P_{jA} P_{AB} + (v-u) P_{jA} P_{jB} \leftarrow P_{jB} P_{AB}$$

$$= (u_j - v + v - u) P_{jB} P_{AB} + (u_j - u) P_{jA} P_{AB}.$$

$$= (u_j - u)(P_{jB} + P_{Aj}) P_{AB} = (u_j - u) P_{AB} (P_{Aj} + P_{Bj})$$ ✓

terms only involving one permutation are the same ✓.

Now we can generalize it

$$R^{AB}_{(u-v)} [ S^{j-1,A}_{(u)} S^{jA}(u) \otimes S^{j-1\,B}(v) S^{jB}(v) ]$$

$$= R^{AB}_{(u-v)} [ S^{j-1A}(u) \otimes S^{j-1\,B}(v).\ S^{jA}(u) \otimes S^{jB}(v) ]$$

$$= [S^{j-1,A}(v) \otimes S^{j-1B}(u)] R^{AB}(u-v) [ S^{jA}(u) \otimes S^{jB}(v) ]$$

$$= [S^{j-1,A}(v) \otimes S^{j-1,B}(u)] [S^{j,A}(v) \otimes S^{jB}(u)] R^{AB}(u-v)$$

$$= [S^{j-1A}(v) S^{jA}(v) \otimes S^{j-1B}(u) S^{jB}(u)] R^{AB}(u-v)$$

$$\Rightarrow R^{AB}(u-v) [ S^{1A}(u) \cdots S^{NA}(u) \otimes S^{1B}(v) \cdots S^{NB}(v) ]$$

$$= [ S^{1A}(v) \cdots S^{NA}(v) \otimes S^{1B}(u) \cdots S^{NB}(u) ] R^{AB}(u-v)$$

RTT

i.e. $\boxed{R^{AB}(u-v)\, T^A(u) \otimes T^B(v) = T^A(v) \otimes T^B(u)\ R^{AB}(u-v)}$

$R^{AB}$ is a 4x4 matrix in the $A \otimes B$ space

$$R^{AB}(u) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c(u) & b(u) & 0 \\ 0 & b(u) & c(u) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and $T^A(u)$ is a $2\times 2$ matrix in the A space

$$T(u) = \begin{pmatrix} A(u), & B(u) \\ C(u) & D(u) \end{pmatrix}$$ and A, B, C, D are $2^N \times 2^N$ matrices in the physical Hilbert space

Based on the "RTT" relation, we have quite a few usefully relations among operators $A(u)$, $B(u)$, $C(u)$ and $D(u)$.

① $A(u)\,A(v) = A(v)\,A(u)$

② $A(u)\,B(v) = c(u-v)\,A(v)B(u) + b(u-v)\,B(v)A(u)$

③ $B(u)\,A(v) = c(u-v)\,B(v)A(u) + b(u-v)\,A(v)\,B(u)$

④ $B(u)\,B(v) = B(v)\,B(u)$

⑤ $A(v)\,B(u) = c(u-v)\,A(u)\,B(v) + b(u-v)\,B(u)\,A(v)$
the same as ②

⑥ $c(u-v)\left[A(u)D(v) - A(v)D(u)\right] = b(u-v)\left[B(v)C(u) - C(u)B(v)\right]$

⑦ $c(u-v)\left[B(u)C(v) - B(v)C(u)\right] = b(u-v)\left[A(v)D(u) - D(u)A(v)\right]$

⑧ $B(v)D(u) = c(u-v)\,B(u)D(v) + b(u-v)\,D(u)\,B(v)$

(9) $$C(v)A(u) = c(u-v)\,C(u)\,A(v) + b(u-v)\,A(u)\,C(v)$$

(10) $$c(u-v)\left[C(u)\,B(v) - C(v)\,B(u)\right] = b(u-v)\left[D(v)A(u) - A(u)D(v)\right]$$

(11) $$c(u-v)\left[D(u)A(v) - D(v)A(u)\right] = b(u-v)\left[C(v)\,B(u) - B(u)\,C(v)\right]$$

(12) $$D(v)\,B(u) = c(u-v)\,D(u)\,B(v) + b(u-v)\,B(u)\,D(v)$$

(13) $$C(u)\,C(v) = C(v)\,C(u)$$

(14) $$C(u)\,D(v) = c(u-v)\,C(v)\,D(u) + b(u-v)\,D(v)\,C(u)$$

(15) $$D(u)\,C(v) = c(u-v)\,D(v)\,C(u) + b(u-v)\,C(v)\,D(u)$$

(16) $$D(u)\,D(v) = D(v)\,D(u)$$

The proof is straight forward from the RTT relation.

$$R^{AB}(u-v)\,T^A(u) \otimes T^B(v) = T^A(v) \otimes T^B(u)\,R^{AB}(u-v)$$

which is a $4\times4$ matrix identity in the $A\otimes B$ space. Expand it and check term by ~~te~~rm. See Appendix for proof. Now we will use it to prove

$$\boxed{\left[\mathrm{tr}_A T(u),\ \mathrm{tr}_A T(v)\right] = 0\ .}$$

Proof: $tr_A T(u) = A(u) + D(u)$, $tr\, T(v) = A(v) + D(v)$

$$[tr_A T(u), tr_A T(v)] = [A(u)+D(u), A(v)+D(v)]$$

$$= [A(u), A(v)] + [D(u), D(v)] + (A(u)D(v) - A(v)D(u) + D(u)A(v) - D(v)A(u)]$$

From relation ① $\Rightarrow [A(u), A(v)] = 0$

⑯ $\Rightarrow [D(u), D(v)] = 0$

⑥ $\Rightarrow A(u)D(v) - A(v)D(u) = \frac{b(u-v)}{c(u-v)}[B(v)C(u) - C(u)B(v)]$

⑪ exchang $u$ and $v$

$$D(v)A(u) - D(u)A(v) = \frac{b(v-u)}{c(v-u)}[C(u)B(v) - B(v)C(u)]$$

since $b(x)/c(x) = -x/i = ix \Rightarrow \frac{b(u-v)}{c(u-v)} = -\frac{b(v-u)}{c(v-u)}$

$$\Rightarrow A(u)D(v) - A(v)D(u) = D(v)A(u) - D(u)A(v)$$

$$\Rightarrow [tr_A T(u), tr_A T(v)] = 0.$$

★ In the A-space, $S^{jA}(u) = b(u-u_j) + c(u-u_j)\frac{1}{2}(1 + \vec{\sigma}_j \cdot \vec{\tau}^A)$

$$= b(u-u_j) + \frac{c(u-u_j)}{2}\Big[(1+\sigma_j^z)\frac{\tau_A^z+1}{2} + (1-\sigma_j^z)\frac{1-\tau_A^z}{2} + 2\sigma_+\frac{\tau_x - i\tau_y}{2} + 2\sigma_-\frac{\tau_x+i\tau_y}{2}\Big]$$

$$\boxed{S^{jA}(u) = b(u-u_j) + c(u-u_j)\begin{bmatrix} \frac{1+\sigma_j^z}{2} & \sigma_- \\ \sigma_+ & \frac{1-\sigma_j^z}{2} \end{bmatrix}}$$

In the Physical Hilbert space, $H_1 \otimes H_2 \cdots \otimes H_n$, we define the vacuum state $|0\rangle = |\uparrow\uparrow\cdots\uparrow\rangle$.

Apply $S^{jA}(u)$ on $|0\rangle$, we have

$$S^{jA}(u)|0\rangle = \begin{bmatrix} b(u-u_j)+c(u-u_j) & c(u-u_j)\sigma_- \\ c(u-u_j)\sigma_+ & b(u-u_j) \end{bmatrix} |0\rangle$$

$$= \begin{bmatrix} 1 & c(u-u_j)\sigma_-^j \\ 0 & b(u-u_j) \end{bmatrix} |0\rangle \,,$$

we have used $b(u-u_j)+c(u-u_j)=1$

$$\Rightarrow T(u)|0\rangle = S^{1A}(u)\, S^{2A}(u) \cdots S^{NA}(u)\; |0\rangle$$

$$= \begin{bmatrix} 1 & c(u-u_1)\sigma_-^1 \\ 0 & b(u-u_1) \end{bmatrix} \begin{bmatrix} 1 & c(u-u_2)\sigma_-^2 \\ 0 & b(u-u_2) \end{bmatrix} \cdots \begin{bmatrix} 1 & c(u-u_N)\sigma_-^N \\ 0 & b(u-u_N) \end{bmatrix} |0\rangle$$

$$= \begin{pmatrix} 1 & B(u) \\ 0 & \prod_{j=1}^{N} b(u-u_j) \end{pmatrix} |0\rangle$$

← complicated

→ can be proved by induction.

$$\boxed{T(u)|0\rangle = \begin{pmatrix} 1 & B(u) \\ 0 & \prod_{j=1}^{N} b(u-u_j) \end{pmatrix} |0\rangle}$$

Hence

$$A(u)|0\rangle = |0\rangle$$

$$D(u)|0\rangle = \prod_{j=1}^{N} b(u-u_j)\, |0\rangle$$

$$C(u)|0\rangle = 0$$

$$B(u)|0\rangle = \sum_j \# |\uparrow\cdots\uparrow\underset{j\text{th}}{\downarrow}\uparrow\cdots\rangle$$

← one magnon state.

✲ $|0\rangle$ is the eigenstate of $tr_A(T(u)) = A(u) + D(u)$ with the eigenvalue $1 + \prod_{j=1}^{N} b(u-u_j)$.

✲ $B(u)$ operator behaves like $S^-$ to creat "magnon" excitation.

To get all the other eigenstates of $tr_A(T(u))$, we apply the flipping operator $B$ on the "ferro" state:

$\underline{B(v_1)\, B(v_2) \cdots B(v_M)\, |0\rangle \text{ is } N-M \text{ spin-up, } M \text{ spin-down.}}$

we prove that it is an eigenstate of $tr\, T(u)$.

From relation ④ $\Rightarrow$ $B(u)\,B(v) = B(v)\,B(u)$ ← commutate

$$\boxed{\begin{aligned} &⑧ \Rightarrow D(u)\,B(v) = \frac{1}{b(u-v)} B(v)\,D(u) - \frac{c(u-v)}{b(u-v)} B(u)\,D(v) \\ &③ \text{ exchange } u \leftrightarrow v \\ &A(u)\,B(v) = \frac{1}{b(v-u)} B(v)\,A(u) - \frac{c(v-u)}{b(v-u)} B(u)\,A(v) \end{aligned}}$$

**Check:** the case that $M=2$

$$A(u)\,B(v_1)\,B(v_2) = \left[\frac{B(v_1)\,A(u)}{b(v_1-u)} - \frac{c(v_1-u)}{b(v_1-u)} B(u)\,A(v_1)\right] B(v_2)$$

$$= \left\{\frac{1}{\underline{b(v_1-u)}} B(v_1)\left[\frac{1}{\underline{b(v_2-u)}} B(v_2)\,A(u) - \frac{c(v_2-u)}{b(v_2-u)} B(u)\,A(v_2)\right]\right.$$
$$\left. - \frac{c(v_1-u)}{b(v_1-u)} B(u)\left[\frac{1}{b(v_2-v_1)} B(v_2)\,A(v_1) - \frac{c(v_2-v_1)}{b(v_2-v_1)} B(v_1)\,A(v_2)\right]\right\}$$

$$D(u)\,B(v_1) \cdots B(v_M) = \left[\frac{1}{b(u-v_1)} B(v_1)\,D(u) - \frac{c(u-v_1)}{b(u-v_1)} B(u)\,D(v_1)\right] B(v_2)$$

$$= \left\{\frac{1}{\underline{b(u-v_1)}} B(v_1)\left[\frac{1}{\underline{b(u-v_2)}} B(v_2)\,D(u) - \frac{c(u-v_2)}{b(u-v_2)} B(u)\,D(v_2)\right]\right.$$
$$- \frac{c(u-v_1)}{b(u-v_1)} B(u)\left[\frac{1}{b(v_1-v_2)} B(v_2)\,D(v_1) - \frac{c(v_1-v_2)}{b(v_1-v_2)} B(v_1)\,D(v_2)\right]$$

$A(u)|0\rangle = |0\rangle$

$D(u)|0\rangle = \prod_{j=1}^{N} b(u-u_j)|0\rangle$

$$[A(u)+D(u)]\, B(v_1)\, B(v_2)\, |0\rangle$$

$$= \left\{ \frac{1}{b(v_1-u)\, b(v_2-u)} + \frac{\prod_{j=1}^{N} b(u-u_j)}{b(u-v_1)\, b(u-v_2)} \right\} B(v_1) B(v_2)\, |0\rangle$$

+ unwanted terms.

If unwanted terms were zero, then $B(v_1)\cdots B(v_M)|0\rangle$ is $tr_A T(u)$'s eigenstate with the eigenvalue $\frac{1}{b(v_1-u)\, b(v_2-u)} + \frac{\prod_{j=1}^{N} b(u-u_j)}{b(u-v_1)\, b(u-v_2)}$

Set $u = u_j \Rightarrow tr_A T(u_j)$'s eigenvalue $\frac{1}{b(v_1-u_j)\, b(v_2-u_j)}$ since $b(0)=0$.

Then we have
$$\boxed{e^{ik_j L} = \frac{1}{b(v_1-u_j)\, b(v_2-u_j)}}$$

the 1st order unwanted terms

$$-\frac{c(v_1-u)}{b(v_1-u)} \left[ \frac{1}{b(v_2-v_1)} - \frac{\prod_{j=1}^{N} b(v_1-u_j)}{b(v_1-v_2)} \right] B(u)\, B(v_2)\, |0\rangle$$

set $\prod_{j=1}^{N} b(v_1-u_j) = \frac{b(v_1-v_2)}{b(v_2-v_1)} = \frac{\prod_{\alpha\neq 1} b(v_1-v_\alpha)}{\prod_{\alpha\neq 1} b(v_\alpha-v_1)} \Rightarrow$ this term vanish

Check terms for $B(v_1)B(u)|0\rangle$ — the calculation is complicated

but since $B(v_1)B(v_2) = B(v_2)B(v_1)$

we should arrive at

$$-\frac{c(v_2-u)}{b(v_2-u)} \left[ \frac{1}{b(v_1-v_2)} - \frac{\prod_{j=1}^{N} b(v_2-u_j)}{b(v_2-v_1)} \right] B(u)\, B(v_1)\, |0\rangle$$

set $$\prod_{j=1}^{N} b(v_2 - u_j) = \frac{b(v_2 - v_1)}{b(v_1 - v_2)} = \frac{\prod_{\alpha\neq 2} b(v_2 - v_\alpha)}{\prod_{\alpha \neq 2} b(v_\alpha - v_2)}$$

Then we can generalize to M - "magon" case

$$\left[A(u) + D(u)\right] B(v_1) B(v_2) \cdots B(v_M) |0\rangle$$

$$= \left\{ \prod_{\alpha=1}^{M} \frac{1}{b(v_\alpha - u)} + \frac{\prod_{j=1}^{N} b(u-u_j)}{\prod_{\alpha=1}^{M} b(u - v_\alpha)} \right\} B(v_1) \cdots B(v_M) |0\rangle$$

$$- \frac{c(v_1 - u)}{b(v_1 - u)} \left\{ \frac{1 \leftarrow A(v_1)|0\rangle}{\prod_{\alpha=2}^{M} b(v_\alpha - v_1)} - \frac{\prod_{j=1}^{N} b(v_1 - u_j) \leftarrow D(v_1)|0\rangle}{\prod_{\alpha=2}^{M} b(v_1 - v_\alpha)} \right\} B(u) B(v_2) \cdots B(v_M) |0\rangle$$

$$- \frac{c(v_k - u)}{b(v_k - u)} \left\{ \frac{1 \leftarrow A(v_k)|0\rangle}{\prod_{\alpha\neq k}^{M} b(v_\alpha - v_k)} - \frac{\prod_{j=1}^{N} b(v_k - u_j) \leftarrow D(v_k)|0\rangle}{\prod_{\alpha \neq k}^{M} b(v_k - v_\alpha)} \right\} B(v_1) \cdots B(u) B(v_{k+1}) \cdots B(v_M) |0\rangle$$

* Due to the permutation symmetry

Hence, we set

$$\boxed{\begin{aligned} &\prod_{\alpha=1}^{M} \frac{1}{b(v_\alpha - u_j)} = e^{i k_j L} \quad \text{for } j = 1, 2, \cdots N \\ &\prod_{j=1}^{N} b(v_k - u_j) = \frac{\prod_{\alpha\neq k}^{M} b(v_k - v_\alpha)}{\prod_{\alpha \neq k}^{M} b(v_\alpha - v_k)} \quad \text{for } k = 1, \cdots M \end{aligned}}$$

All unwanted terms vanish, and

$$\mathrm{tr}_A \left[ T(u_j) \right] \left\{ B(v_1) B(v_2) \cdots B(v_M) |0\rangle \right\} = e^{i k_j L} \left[ B(v_1) \cdots B(v_M) \right] |0\rangle.$$

Plug in $u_j = \frac{k_j}{c}$ , $v_\alpha = \frac{\Lambda_\alpha}{c} + \frac{i}{2}$ , $b(u) = \frac{-u}{-u+i}$

$$\Rightarrow \quad b(v_\alpha - u_j) = \frac{-\frac{\Lambda_\alpha}{c} + \frac{k_j}{c} - \frac{i}{2}}{-\frac{\Lambda_\alpha}{c} + \frac{k_j}{c} + \frac{i}{2}} = \frac{k_j - \Lambda_\alpha - \frac{ic}{2}}{k_j - \Lambda_\alpha + \frac{ic}{2}}$$

$$b(v_\alpha - v_\kappa) = \frac{\Lambda_\kappa - \Lambda_\alpha}{\Lambda_\kappa - \Lambda_\alpha + ic},$$

$$b(v_\kappa - v_\alpha) = \frac{\Lambda_\alpha - \Lambda_\kappa}{\Lambda_\alpha - \Lambda_\kappa + ic}$$

$$\Rightarrow \quad \boxed{e^{ik_j L} = \prod_{\alpha=1}^{M} \frac{k_j - \Lambda_\alpha + \frac{ic}{2}}{k_j - \Lambda_\alpha - \frac{ic}{2}}} \leftarrow \text{for each "} k_j \text{" } j=1, \cdots N$$

$$\prod_{j=1}^{N} \frac{(k_j - \Lambda_\kappa) - \frac{ic}{2}}{k_j - \Lambda_\kappa + \frac{ic}{2}} = \prod_{\alpha=1, \alpha \neq \kappa}^{M} \frac{\Lambda_\alpha - \Lambda_\kappa - ic}{\Lambda_\alpha - \Lambda_\kappa + ic}$$

or rewrite

$$\boxed{\prod_{j=1}^{N} \frac{k_j - \Lambda_\alpha + \frac{ic}{2}}{k_j - \Lambda_\alpha - \frac{ic}{2}} = - \prod_{\beta=1}^{M} \frac{\Lambda_\beta - \Lambda_\alpha + ic}{\Lambda_\beta - \Lambda_\alpha - ic} \quad \text{for } \alpha = 1, \cdots M}$$

# Appendix:

$$\left[R^{AB}(u-v)\,T^A(u)\,T^B(v)\right]_{u\omega,\,u'\omega'} = \Big[c(u-v)\,\delta_{u\omega,\,u''\omega''} + b(u-v)\,P_{u\omega,\,u''\omega''}\Big]\overbrace{\underset{u''u'}{T^A(u)}\;\underset{\omega''\omega'}{T^B(v)}}$$

$$= c(u-v)\,T^A_{uu'}(u)\,T^B_{\omega\omega'}(v) + b(u-v)\,T^A_{\omega u'}(u)\,T^B_{u\omega'}(v)$$

$$\left[T^A(v)\,T^B(u)\,R^{AB}(u-v)\right]_{u\omega,\,u'\omega'} = \underset{uu''}{T^A(v)}\;\underset{\omega\omega''}{T^B(u)}\Big[c(u-v)\,\delta_{u''\omega'',\,u'\omega'} + b(u-v)\,P_{u''\omega'',\,u'\omega'}\Big]$$

$$= c(u-v)\,T^A_{uu'}(v)\,T^B_{\omega\omega'}(u) + b(u-v)\,\underset{u\omega'}{T^A(v)}\;\underset{\omega u'}{T^B(u)}$$

$\overset{u\omega\ u'\omega'}{(11,11)}$: $\quad T_{11}(u)\,T_{11}(v) = T_{11}(v)\,T_{11}(u)$, $\Rightarrow$ $\underline{A(u)A(v) = A(v)A(u)}$,

$(11,12)$ $\quad c(u-v)\,T_{11}(u)\,T_{12}(v) + b(u-v)\,T_{11}(u)\,T_{12}(v)$

$$= c(u-v)\,T_{11}(v)\,T_{12}(u) + b(u-v)\,T_{12}(v)\,T_{11}(u)$$

$$\Rightarrow \quad \underline{A(u)B(v) = c(u-v)\,A(v)\,B(u) + b(u-v)\,B(v)\,A(u)}$$

$(11, \overset{21}{\cancel{13}})$ $\quad \cancel{c(u-v)\,T_{11}(u)\,T_{13}(v) + b(u-v)\,T_{11}(u)\,T_{13}(v)}$

$$\cancel{= c(u-v)\,T_{11}(v)\,T_{13}(u) + b(u-v)\,T_{13}(v)\,T_{11}(u)}$$

$$\Rightarrow \quad c(u-v)\,T_{12}(u)\,T_{11}(v) + b(u-v)\,T_{12}(u)\,T_{11}(v)$$

$$= c(u-v)\,T_{12}(v)\,T_{11}(u) + b(u-v)\,T_{11}(v)\,T_{12}(u)$$

$$\underline{B(u)A(v) = c(u-v)\,B(v)\,A(u) + b(u-v)\,A(v)\,B(u)}$$

$$\Rightarrow \quad A(v)B(u) = \frac{1}{b(u-v)}\,B(u)A(v) - \frac{c(u-v)}{b(u-v)}\,B(v)A(u)$$

$uw\ u'w'$ $(11, 22)$

$$\left.\begin{aligned} &c(u-v)\, T_{12}(u)\, T_{12}(v) + b(u-v)\, T_{12}(u)\, T_{12}(v) \\ &= c(u-v)\, T_{12}(v)\, T_{12}(u) + b(u-v)\, T_{12}(v)\, T_{12}(u) \end{aligned}\right\} \Rightarrow \boxed{B(u)\,B(v) = B(v)\,B(u)}$$

$uw\ u'w'$ $(12, 11)$

$$c(u-v)\, T_{11}(u)\, T_{21}(v) + b(u-v)\, T_{21}(u)\, T_{11}(v)$$

$$= c(u-v)\, T_{11}(v)\, T_{21}(u) + b(u-v)\, T_{11}(v)\, T_{21}(u)$$

$$\underline{c(u-v)\, A(u)\, B(v) + b(u-v)\, B(u)\, A(v) = A(v)\, B(u)}$$

$uw\ u'w'$ $(12, 12)$

$$c(u-v)\, T_{11}(u)\, T_{22}(v) + b(u-v)\, T_{21}(u)\, T_{12}(v)$$

$$= c(u-v)\, T_{11}(v)\, T_{22}(u) + b(u-v)\, T_{12}(v)\, T_{21}(u)$$

$$\star\ \underline{c(u-v)\, A(u) D(v) + b(u-v)\, C(u)\, B(v) = c(u-v)\, A(v)\, D(u) + b(u-v)\, B(v)\, C(u)}$$

$uw\ u'w'$ $12\ 21$

$$c(u-v)\, T_{12}(u)\, T_{21}(v) + b(u-v)\, T_{22}(u)\, T_{11}(v)$$

$$= c(u-v)\, T_{12}(v)\, T_{21}(u) + b(u-v)\, T_{11}(v)\, T_{22}(u)$$

$$\star\ \underline{c(u-v)\, B(u)\, C(v) + b(u-v)\, D(u)\, A(v)}$$

$$\underline{= c(u-v)\, B(v)\, C(u) + b(u-v)\, A(v)\, D(u)}$$

$uw, u'w'$ $12\ 22$

$$c(u-v)\, T_{12}(u)\, T_{22}(v) + b(u-v)\, T_{22}(u)\, T_{12}(v)$$

$$= c(u-v)\, T_{12}(v)\, T_{22}(u) + b(u-v)\, T_{12}(v)\, T_{22}(u)$$

$$\underline{c(u-v)\, B(u)\, D(v) + b(u-v)\, D(u)\, B(v)} \Rightarrow \boxed{D(u)\,B(v) = \frac{B(v)\,D(u)}{b(u-v)} - \frac{c(u-v)}{b(u-v)}\, B(u)\, D(v)}$$

$$= \cancel{c(u-v)}\ \underline{B(v)\, D(u)}$$

$uw \; u'w'$
$21 \; 11$

$$c(u-v)\, T_{21}(u)\, T_{11}(v) + b(u-v)\, T_{11}(u)\, T_{21}(v)$$

$$= c(u-v)\, T_{21}(v)\, T_{11}(u) + b(u-v)\, T_{21}(v)\, T_{11}(u)$$

$$c(u-v)\, \mathbb{C}(u)\, A(v) + b(u-v)\, A(u)\, \mathbb{C}(v) = \mathbb{C}(v)\, A(u)$$

$21 \; 12$

$$c(u-v)\, T_{21}(u)\, T_{12}(v) + b(u-v)\, T_{11}(u)\, T_{22}(v)$$

$$= c(u-v)\, T_{21}(v)\, T_{12}(u) + b(u-v)\, T_{22}(v)\, T_{11}(u)$$

$$c(u-v)\, \mathbb{C}(u)\, B(v) + b(u-v)\, A(u)\, D(v) = c(u-v)\, \mathbb{C}(v)\, B(u) + b(u-v)\, D(v)\, A(u)$$

$21 \; 21$

$$\cancel{c(u)}\; c(u-v)\, T_{22}(u)\, T_{11}(v) + b(u-v)\, T_{12}(u)\, T_{21}(v)$$

$$= c(u-v)\, T_{22}(v)\, T_{11}(u) + b(u-v)\, T_{21}(v)\, T_{12}(u)$$

$$c(u-v)\, D(u)\, A(v) + b(u-v)\, B(u)\, C(v) = c(u-v)\, D(v)\, A(u) + b(u-v)\, C(v)\, B(u)$$

$21 \; 22$

$$c(u-v)\, T_{22}(u)\, T_{12}(v) + b(u-v)\, T_{12}(u)\, T_{22}(v)$$

$$= c(u-v)\, T_{22}(v)\, T_{12}(u) + b(u-v)\, T_{22}(v)\, T_{12}(u)$$

$$c(u-v)\, D(u)\, B(v) + b(u-v)\, B(u)\, D(v) = c(u-v)\, D(v)\, B(u) + b(u-v)\, D(v)\, B(u)$$

$$\boxed{= D(v)\, B(u)}$$

$22 \; 11,$

$$c(u-v)\, T_{21}(u)\, T_{21}(v) + b(u-v)\, T_{21}(u)\, T_{21}(v)$$

$$= c(u-v)\, T_{21}(v)\, T_{21}(u) + b(u-v)\, T_{21}(v)\, T_{21}(u)$$

$$\cancel{c(u-v)}\; \boxed{C(u)\, C(v) = C(v)\, C(u)}$$

$uw\ u'w'$

$22,12$ $\quad c(u-v)\, T_{21}(u) T_{22}(v) + b(u-v)\, T_{21}(u)\, T_{22}(v)$ 

$= c(u-v)\, T_{21}(v)\, T_{22}(u) + b(u-v)\, T_{22}(v)\, T_{21}(u)$

$\Rightarrow \underline{C(u) D(v) = c(u-v)\, C(v)\, D(u) + b(u-v)\, D(v)\, C(u)}$

$uw\ u'w'$

$22\ 21$ $\quad c(u-v)\, T_{22}(u) T_{21}(v) + b(u-v)\, T_{22}(u)\, T_{21}(v)$

$= c(u-v)\, T_{22}(v) T_{21}(u) + b(u-v)\, T_{21}(v)\, T_{22}(u)$

$\underline{D(u)\, C(v) = c(u-v)\, D(v)\, C(u) + b(u-v)\, C(v) D(u)}$

$22\ 22 \quad \Rightarrow \quad \underline{D(u)\, D(v) = D(v)\, D(u)}$

Next we prove:

$$S^+ \, B(v_1) \cdots B(v_m) \, |0\rangle ,$$

$$S^3 \, B(v_1) \cdots B(v_m) \, |0\rangle = \frac{N-2m}{2} \, B(v_1) \cdots B(v_m) \, |0\rangle ,$$

where $S^\pm = S^1 \pm i S^2$, i.e. $B(v_1)\cdots B(v_m)$ is $S_{tot} = \frac{N-2m}{2}$ state.

Proof: $[\, S^{j,A}(u), S_j^n \,] = [\, b(u-u_j) + c(u-u_j) \frac{1+\vec{\tau}\cdot\vec{\sigma}_j}{2} , \frac{\sigma_j^n}{2} \,]$ (n=1,2,3 spin orientation)

$$= \frac{1}{4} c(u-u_j) \;\; \tau^m \, [\, \sigma_j^m \; \sigma_j^n \,] \quad 2i\epsilon^{mnl} \, \tau^m \sigma^l$$

$$\parallel$$

$$= -\frac{1}{4} c(u-u_j) \;\; \sigma_j^l \, [\, \tau_j^l \; \tau^n \,] \quad 2i\epsilon^{lnm} \, \sigma^l \tau^m$$

$$= \boxed{-\,[\, S^{j,A}(u) , \tfrac{1}{2}\tau^n \,] = [\, S^{j,A}(u) , S_j^n \,]}$$

$$T(u) = S^{1A}(u) \, S^{2A}(u) \cdots S^{NA}(u)$$

$$[\, T(u), S_{tot}^n \,] = \sum_{j=1}^{N} S^{1A} \cdots [\, S^{jA} , S_{tot}^n \,] \cdots S^{NA}$$

$$= -\sum_j L_1 \cdots [\, L_j , \tfrac{\tau^n}{2} \,] \cdots L_N = -[\, T(u) , \tfrac{1}{2}\tau^n \,]$$

in the A space: $T_{\alpha\beta}(u)$ ← explicitly write the matrix element.

$$\Rightarrow [\, T_{\alpha\beta}(u) , S_{tot}^n \,] = -[\, T(u) \, \tfrac{1}{2}\tau^n \,] = \tfrac{1}{2} [\, \tau^n_{\alpha\beta'} T_{\beta'\beta} - T_{\alpha\beta'} \tau^n_{\beta'\beta} \,]$$

$$T_N = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

$\Rightarrow$

$$\Rightarrow \begin{pmatrix} [A, S^n], & [B, S^n] \\ [C, S^n], & [D, S^n] \end{pmatrix} = -\left[ \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \ \frac{1}{2}\tau^n \right]$$

$$-\frac{1}{2}\left[ \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \right] = \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}, \quad -\frac{1}{2}\left[ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \right] = -\frac{1}{2}\begin{pmatrix} B-C, & A-D \\ D-A, & C-B \end{pmatrix}$$

$$-\frac{1}{2}\left[ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} & -i \\ i & \end{pmatrix} \right] = -\frac{1}{2}\begin{bmatrix} (B+C)i, & -(A-D)i \\ (D-A)i, & -(B+C)i \end{bmatrix}$$

$$\Rightarrow [A+D, S^n] = 0, \quad \text{for } n = 1, 2, 3$$

$$[B(u)\ S^3] = B(u), \quad [B(u), S^+] = [B(u), S^1] + i[B(u), S^2]$$

$$= -\frac{1}{2}(A-D) - \frac{1}{2}(A-D) = -(A-D)$$

or rewrite $\boxed{[S^n, A+D] = 0, \ [S^3, B(u)] = -B(u), \ [S^+, B(u)] = A(u) - D(u)}$

Obviously $\quad S^+|0\rangle = 0, \quad S^3|0\rangle = \frac{N}{2}|0\rangle$

$$S^3 B(u) = B(u)(S^3 - 1) \Rightarrow$$

$$S^3 B(v_1) \cdots B(v_m)\ |0\rangle = B(v_1)(S^3 - 1) B(v_2) \cdots \ |0\rangle$$

$$= B(v_1) B(v_2)(S^3 - 2) \cdots |0\rangle = B(v_1) \cdots B(v_m)(S^3 - M)|0\rangle$$

$$= \frac{N - 2M}{2} B(v_1) \cdots B(v_m)|0\rangle$$

$$[S^+, B(v_1)\cdots B(v_m)] = \sum_\alpha B(v_1)\cdots [S^+, B(v_\alpha)]\cdots B(v_m)$$

$$= \sum_\alpha B(v_1)\cdots (A(v_\alpha) - D(v_\alpha))\cdots B(v_m)$$

$$\Rightarrow \quad S^+ B(v_1)\cdots B(v_m)|0\rangle = [S^+, B(v_1)\cdots B(v_m)]|0\rangle = \sum_\alpha B(v_1)\cdots(A(v_\alpha)-D(v_\alpha))\cdots B(v_m)|0\rangle$$

by using
$$A(u)B(v) = \frac{1}{b(v-u)} B(v)A(u) - \frac{c(v-u)}{b(v-u)} B(u)A(v)$$

$$D(u)B(v) = \frac{1}{b(u-v)} B(v)D(u) - \frac{c(u-v)}{b(u-v)} B(u)D(v)$$

$$D(u)|0\rangle = \prod_{j=1}^{N} b(u-u_j)|0\rangle, \quad A|0\rangle = |0\rangle.$$

We can expect that, the finial expression can be expressed as

$$S^+ B(v_1)\cdots B(v_m)|0\rangle = \sum_\alpha M_\alpha B(v_1)\cdots B(v_{\alpha-1}) B(v_{\alpha+1})\cdots B(v_m)|0\rangle$$

The coefficient of $M_1$ can be obtained as — the term missing $\boxed{B(v_1)}$

$$(A(v_1) - D(v_1)) B(v_2)\cdots B(v_m)|0\rangle$$

$$\vec{M}_1 = \prod_{\alpha=2}^{M} \frac{1}{b(v_\alpha - v_1)} - \frac{\prod_{j=1}^{N} b(v_1 - u_j)}{\prod_{\alpha=2}^{M} b(v_1 - v_2)} = 0. \leftarrow \text{Bethe antsatz Eq.}$$

Because all B commutate, we can do arbitary permutation of B. Thus from $M_1 = 0$, we obtain $M_\alpha = 0$.

$$\Rightarrow \boxed{S^+ B(v_1)\cdots B(v_m)|0\rangle = 0}$$