2. δ-interaction of boson

$$H = \int dx \left(\frac{\partial u^+}{\partial x} \frac{\partial u}{\partial x} + c\, u^+ u^+ u\, u \right)$$ second Quantization

$$[\, u(x,t),\ u^+(x,t) \,] = \delta(x-x')$$

set $|\psi\rangle = \int dx_1 \cdots dx_N \ \psi(x_1 \cdots x_N, t)\ u^+(x_1) \cdots u^+(x_N)\, |0\rangle$

$\rightarrow$ δ interaction

$$\boxed{ -\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} \psi + 2c \sum_{i<j} \delta(x_i - x_j)\, \psi = E\, \psi }$$ first Quantizatio

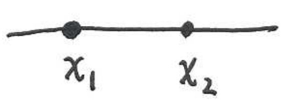
Assume $c > 0$, and looking for Bethe ansatz solution.

example of $N=2$

$$-\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right) \psi + 2c\, \delta(x_1 - x_2)\, \psi = E\, \psi$$

ψ is continous but derivtive is discontinous in

$$\psi = \Theta(x_2 - x_1)\, \psi_{12}(x_1, x_2) + \Theta(x_1 - x_2)\, \psi_{21}(x_1, x_2)$$



Bose statistics $\quad \psi(x_1\, x_2) = \psi(x_2\, x_1)$

$$\Rightarrow \psi_{12}(x_1, x_2) = \psi_{21}(x_2, x_1)$$

For $x_1 < x_2$, $\psi = A_{12}\, e^{ik_1x_1 + ik_2x_2} + A_{21}\, e^{ik_2x_1 + ik_1x_2}$

($k_1 \neq k_2$, you can prove if $k_1 = k_2$, then $\psi \equiv 0$)

if $x_2 < x_1$, $\psi = A_{12}\, e^{ik_2x_1 + ik_1x_2} + A_{21}\, e^{ik_1x_1 + ik_2x_2}$

set relative coordinate / center of mass coordinate

$$y = x_2 - x_1, \quad X = \frac{x_1 + x_2}{2}, \quad K = k_1 + k_2$$

$$\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} = \frac{1}{2}\frac{\partial^2}{\partial X^2} + 2\frac{\partial^2}{\partial y^2}$$

$$\psi = \begin{cases} e^{iKX}\left(A_{12}\, e^{i(k_2-k_1)y/2} + A_{21}\, e^{-i(k_2-k_1)y/2}\right) & y > 0 \\ e^{iKX}\left(A_{12}\, e^{-i(k_2-k_1)y/2} + A_{21}\, e^{i(k_2-k_1)y/2}\right) & y < 0 \end{cases}$$

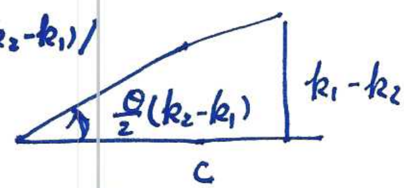
in the center of mass coordinate

$$-2\frac{\partial^2}{\partial y^2}\psi + 2c\,\delta(y)\,\psi = E\psi$$

$$\rightarrow \quad \left.\frac{\partial\psi}{\partial y}\right|_{0^+} - \left.\frac{\partial\psi}{\partial y}\right|_{0^-} = c\,\psi\big|_0$$

$$\Rightarrow \quad i(k_2 - k_1)(A_{12} - A_{21}) = c\,(A_{12} + A_{21})$$

$$\Rightarrow \quad \frac{A_{21}}{A_{12}} = -\frac{c + i(k_1 - k_2)}{c - i(k_1 - k_2)} = -e^{i\theta(k_2-k_1)}$$



$$\frac{1}{2}\theta(k_2 - k_1) = \tan^{-1}\frac{k_1 - k_2}{2} \quad \text{or} \quad \boxed{\theta(k_2 - k_1) = -2\tan^{-1}\frac{k_2 - k_1}{c}}$$

Now we consider $N \geq 3$, define permutation $Q = (Q_1, Q_2, \cdots Q_N)$

$$\boxed{\psi = \sum_Q \theta(x_{Q_1} < x_{Q_2} < \cdots x_{Q_N})\, \psi_Q(x_1, \cdots x_N)}$$

Boson statistics $\longrightarrow$ $\psi_{Q=E}(x_{Q_1} \cdots x_{Q_N}) = \psi_Q(x_1 \cdots x_N)$

if we know $\psi_{12\cdots N}(x_1 \cdots x_N)$, then we know everything.

For example $\psi_{23\cdots N1}(x_1, \cdots x_N) = \psi_{12\cdots N}(x_2\, x_3 \cdots x_N, x_1)$

Bethe ansatz: for $x_1 < x_2 < \cdots < x_N$

$$\psi = \psi_{12\cdots N} = \sum_P A_P\, e^{ik_{P_1}x_1 + ik_{P_2}x_2 + \cdots ik_{P_N}x_N},$$

where $P = (P_1, P_2, \cdots P_N)$, $k_1 \cdots k_N$ are unequal real numbers.

In the region of $x_2 < \cdots < x_N < x_1$

$$\psi = \psi_{23\cdots N1} = \sum_P A_P\, e^{ik_{P_1}x_2 + ik_{P_2}x_3 + \cdots + ik_{P_N}x_1}.$$

Similarly to case of $N = 2$, for the two permutation

$$\begin{array}{ll} & \qquad\quad j \quad j+1 \\ k_{P_1}\, k_{P_2} \cdots k_{P_N} = & \cdots\ k \quad k' \cdots \\ k_{P_1'}\, k_{P_2'} \cdots k_{P_N'} = & \cdots\ k' \quad k \cdots \end{array}$$

i.e $(P_1\, P_2 \cdots P_N) = (j, j+1)(P_1'\, P_2' \cdots P_N')$

Consider
$$\psi_{123\cdots N} = \sum_P{}' \Big\{ A_P\, e^{ik_{P_1}x_1 + ik_{P_2}x_2 + \cdots \underline{ik_{P_j}x_j + ik_{P_{j+1}}x_{j+1}} + \cdots} + A_{P'}\, e^{ik_{P_1}x_1 + ik_{P_2}x_2 + \cdots \underline{ik_{P_{j+1}}x_j + ik_{P_j}x_{j+1}} + \cdots} \Big\}$$

$$\psi_{123\cdots j+1, j\cdots N} = \sum_P{}' \Big[A_P\, e^{ik_{P_1}x_1 + \cdots + \underline{ik_{P_j}x_{j+1} + ik_{P_{j+1}}x_j} + \cdots} + A_{P'}\, e^{ik_{P_1}x_1 + \cdots \underline{ik_{P_{j+1}}x_{j+1} + ik_{P_j}x_j} + \cdots} \Big]$$

introducing the collective coordinate between x_j and x_{j+1} as X and y

$$\psi = \begin{cases} \sum_P \cancel{A}\, e^{ik_{P_1}x_1 + \cdots} \left[A_P\, e^{i(k_{P_{j+1}} - k_{P_j})y/2} + A_{P'}\, e^{-i(k_{P_{j+1}} - k_{P_j})y/2} \right] e^{iKX} & (y>0) \\ \sum_P e^{ik_{P_1}x_1 + \cdots} \left[A_P\, e^{-i(k_{P_{j+1}} - k_{P_j})y/2} + A_{P'}\, e^{i(k_{P_{j+1}} - k_{P_j})y/2} \right] e^{iKX} & (y<0) \end{cases}$$

The continuity relation

$$\frac{\partial \psi(x_1 \cdots y)}{\partial y}\bigg|_{0^+} - \frac{\partial \psi(x_1 \cdots y, X \cdots)}{\partial y}\bigg|_{0^-} = c\, \psi(x_1 \cdots y, X)\Big|_{y=0}$$

$\Rightarrow$ For each P and its partner $P' = (j, j+1)P$, we ~~ne~~ need

$$i(k_{P_{j+1}} - k_{P_j})(A_P - A_{P'}) = c(A_P + A_{P'})$$

$$\Rightarrow \boxed{\frac{A_{P'}}{A_P} = -\frac{c + i(k_{P_{j+1}} - k_{P_j})}{c - i(k_{P_{j+1}} - k_{P_j})} = -e^{i\theta(k_{P_{j+1}} - k_{P_j})} \quad \text{and} \quad \theta(k_{P_{j+1}} - k_{P_j}) = -2\tan^{-1}\frac{k_{P_{j+1}} - k_{P_j}}{c}.}$$

Let us consider periodic boundry condition:

$$\psi(x_1=0, x_2 \cdots x_N) = \psi(L, x_2, \cdots x_N)$$

where $x_1 < x_2 < \cdots x_N$ $\qquad\qquad$ $x_2 < x_3 < \cdots < x_N < x_1$

$$\psi_{2\cdots N1}(x_1\, x_2 \cdots x_N) = \sum_P A_{P_1 \cdots P_N}\, e^{ik_{P_1} x_2 + k_{P_2} x_3 + \cdots ik_{P_N} x_1}$$

$$= \sum_P A_{P_2 \cdots P_N P_1}\, e^{ik_{P_2} x_2 + \cdots ik_{P_1} x_1}$$

$$\Rightarrow \psi_{12\cdots N}(0, x_2 \cdots x_N) = \psi_{2\cdots N1}(L, x_2 \cdots x_N)$$

$$\sum_P A_{P_1 \cdots P_N}\, e^{ik_{P_1} + ik_{P_2} x_2 + \cdots} = \sum_P A_{P_2 \cdots P_N P_1}\, e^{ik_{P_1} L + ik_{P_2} x_2 + \cdots}$$

$$\Rightarrow \boxed{A_{P_1 \cdots P_N} = A_{P_2 \cdots P_N P_1}\, e^{ik_{P_1} L}}$$

$\cancel{A_{P_1 \cdots P_N}}$

$$\frac{A_{P_2 P_1 \cdots P_N}}{A_{P_1 P_2 \cdots P_N}} \cdot \frac{A_{P_2 P_3 P_1 \cdots P_N}}{A_{P_2 P_1 P_3 \cdots P_N}} \cdot \frac{A_{P_2 P_3 P_4 P_1 \cdots P_N}}{A_{P_2 P_3 P_1 \cdots P_N}} \cdots \frac{A_{P_2 \cdots P_N P_1}}{A_{P_2 P_3 \cdots P_1 P_N}} = \frac{A_{P_2 \cdots P_N P_1}}{A_{P_1 P_2 \cdots P_N}} = e^{-ik_{P_1} L}$$

$$(-)^{N-1}\, e^{i\theta(k_{P_2} - k_{P_1})}\, e^{i\theta(k_{P_3} - k_{P_1})} \cdots e^{i\theta(k_{P_N} - k_{P_1})} = e^{-ik_{P_1} L}$$

$$\text{or} \quad e^{ik_{P_1} L} = (-)^{N-1}\, e^{+i\sum_{j=1}^{N} \theta(k_{P_1} - k_{P_j})}$$

Set $k_{P_1} = i$

$$\Rightarrow \boxed{e^{ik_i L} = (-)^{N-1}\, e^{+i\sum_{j=1}^{N} \theta(k_i - k_j)}} \quad \leftarrow \text{Bethe ansatz Eq.}$$

$$\boxed{k_i L = 2\pi I_i + \sum_{j=1}^{N} \theta(k_i - k_j), \quad I_i = \begin{cases} \text{integer}, & N = \text{odd} \\ \text{half integer} & N = \text{even} \end{cases}}$$