spinless fermion t−V model

$$H = -t\sum_{\langle ij\rangle}(C_i^{\dagger}C_j + h.c) + V\sum_{\langle i,i+1\rangle} n_i n_{i+1}$$

low energy continous model

$$\frac{1}{a} n_i = J_R + J_L + N^{\dagger} e^{-2ik_f x} + N e^{2ik_f x}$$

where $J_R = \psi_R^{\dagger}\psi_R$, $J_L = \psi_L^{\dagger}\psi_L$

$N^{\dagger} = \psi_R^{\dagger}\psi_L$ $\quad N = \psi_L^{\dagger}\psi_R$

$$\frac{1}{a^2} n_i n_{i+1} = J_R^2 + J_L^2 + 2J_RJ_L + N^{\dagger}N e^{-2ik_f r_i + 2ik_f(r_i+a)}$$
$$+ NN^{\dagger} e^{2ik_f r_i - 2ik_f(r_i+a)} + N^{\dagger}N^{\dagger} e^{-4ik_f r_i - 2ik_f a}$$
$$+ NN e^{2ik_f r_i + 2ik_f r_i + 2ik_f a}$$

$$:N^{\dagger}N: = :\psi_R^{\dagger}\psi_L\psi_L^{\dagger}\psi_R: = -\psi_R^{\dagger}\psi_R\psi_L^{\dagger}\psi_L = -J_RJ_L$$

$$:N^{\dagger}N^{\dagger}: = :\psi_R^{\dagger}\psi_L\psi_R^{\dagger}\psi_L: = :\psi_R^{\dagger}(x)\psi_R^{\dagger}(x+\epsilon)\psi_L(x+\epsilon)\psi_L(x):$$

keep non-chiral part,

$$n_i n_{i+1} = [2 - 2\cos(2k_f)]J_RJ_L + \psi_R^{\dagger}(x)\psi_R^{\dagger}(x+\epsilon)\psi_L(x+\epsilon)\psi_L(x)\cdot e^{\{-i(4k_f-G)\cdot x - 2ik_f\}}$$
$$+ h.c$$

G: reciprocal lattice vector

$$H_0 = \int dx\, v_F \left[\psi_R^\dagger (-i\partial_x) \psi_R + \psi_L^\dagger (i\partial_x) \psi_L \right]$$

$$\left.\boxed{\begin{array}{l} J_R(x) = \sqrt{\frac{1}{\pi}}\, \partial_x \phi_R(x) \\ \psi_R^\dagger (-i\partial_x)\psi_R = (\partial_x \phi_R)^2 \end{array}}\right\} \Rightarrow \psi_R^\dagger(-i\partial_x)\psi_R = \pi :J_R(x) J_R(x):$$

This relation can also be obtained through fermionic OPE.

current algebra form

$$\xrightarrow{\quad} \Rightarrow H_0 = v_F \pi \int dx \left[:J_L^2: + :J_R^2: \right]$$ for spinless fermion

$$H_{int} = \int dx\; 2V[1 - \cos 2k_f] :J_R J_L:$$

$$+ V\left\{ e^{-i(4k_f - G)x - 2ik_f} :\psi_R^\dagger(x)\, \psi_R^\dagger(x+\epsilon)\, \psi_L(x+\epsilon)\, \psi_L(x): \right.$$

$$\left. + h.c. \right\}$$

bosonic

$$\xrightarrow{\quad} J_L^2 + J_R^2 = \frac{1}{\pi}\left[(\partial_x \phi_R)^2 + (\partial_x \phi_L)^2 \right] = \frac{1}{2\pi}\left[(\partial_x \phi)^2 + (\partial_x \Theta)^2 \right]$$

$$J_R J_L = \frac{1}{\pi}\, \partial_x \phi_R\, \partial_x \phi_L = \frac{1}{4\pi}\left[(\partial_x \phi)^2 - (\partial_x \Theta)^2 \right]$$

$$\psi_R^\dagger(x)\psi_R^\dagger(x+\epsilon) = \frac{1}{2\pi a}\, e^{-i\sqrt{4\pi}\,\phi_R(x)}\, e^{-i\sqrt{4\pi}\,\phi_R(x+\epsilon)}$$

$$= \frac{1}{2\pi a}\, e^{-i\sqrt{16\pi}\,\phi_R}\, e^{\frac{(-i)^2}{2}\cdot 4\pi\, [\phi_R(x), \phi_R(x+\epsilon)]}$$

$$= \frac{1}{2\pi a}\, e^{-i\sqrt{16\pi}\,\phi_R}\, e^{-2\pi \frac{i}{4} \mathrm{sgn}(-\epsilon)}$$

$$\psi_L(x+\epsilon)\, \psi_L(x) = \frac{1}{2\pi a}\, e^{-i\sqrt{4\pi}\,\phi_L(x+\epsilon)}\, e^{-i\sqrt{4\pi}\,\phi_L(x)} = \frac{1}{2\pi a}\, e^{-i\sqrt{16\pi}\,\phi_L}\, e^{\frac{(-i)^2}{2}\cdot 4\pi [\phi_L(x+\epsilon), \phi_L]}$$

$$\psi_L(x+\epsilon)\psi_L(x) = \frac{1}{2\pi a} e^{-i\sqrt{16\pi}\,\phi_L} e^{-2\pi\left(-\frac{i}{4}\,sgn(\epsilon)\right)}$$

$$\Rightarrow \psi_R^\dagger(x)\,\psi_R^\dagger(x+\epsilon)\,\psi_L(x+\epsilon)\,\psi_L(x) = \frac{1}{(2\pi a)^2} e^{-i\sqrt{16\pi}\,\phi_R} e^{-i\sqrt{16\pi}\,\phi_L} e^{i\pi\, sgn\,\epsilon}$$

$$= \frac{-1}{(2\pi a)^2} e^{-i\sqrt{16\pi}\,\phi_R} e^{-i\sqrt{16\pi}\,\phi_L} = \frac{-1}{(2\pi a)^2} e^{-i\sqrt{16\pi}\,\phi} e^{\frac{(-i)^2}{2}\cdot 16\pi\,[\phi_R,\phi_L]}$$

$$\downarrow$$

$$= \frac{-1}{(2\pi a)^2} e^{-i\sqrt{16\pi}\,\phi} \qquad e^{-8\pi\cdot\frac{i}{4}} = 1$$

or we can start from $N^\dagger N^\dagger = \left(\frac{i}{2\pi a}\right)^2 e^{-i\sqrt{4\pi}\,\phi(x)} e^{-i\sqrt{4\pi}\,\phi(x+\epsilon)}$

$$= \frac{-1}{(2\pi a)^2} e^{-i\sqrt{16\pi}\,\phi}, \qquad \boxed{[\phi(x),\ \phi(x+\epsilon)] = 0}$$

✓

consistent !

$$\Rightarrow \quad H = \int dx\ \frac{v_F\pi}{2\pi}\left[(\partial_x\phi)^2 + (\partial_x\theta)^2\right] + \frac{2V}{4\pi}[1-\cos 2k_F]\left[(\partial_x\phi)^2 - (\partial_x\theta)^2\right]$$

$$- \frac{2V}{(2\pi a)^2}\cos[\sqrt{16\pi}\,\phi + 2k_F a + \delta x] \qquad \boxed{\delta = 4k_F a - G}$$

$$\frac{\delta}{2} = (2k_F a - \pi)$$

$$\cos 2k_F = -\cos\frac{\delta}{2}$$

$\Rightarrow$

$$H = \frac{v_F}{2}\int dx \left[1+\frac{Va}{\pi v_F}(1+\cos\tfrac{\delta}{2})\right](\partial_x\phi)^2$$
$$+\left[1-\frac{Va}{\pi v_F}(1+\cos\tfrac{\delta}{2})\right](\partial_x\theta)^2$$

$$+\int dx\ \frac{2Va}{(2\pi a)^2}\cos\left[\sqrt{16\pi}\,\phi+\frac{\delta}{2}+\delta x\right]$$

set incommensuribility $\delta=0$

$$\Rightarrow H = \frac{v_E}{2}\int dx\ \left[\frac{1}{K}(\partial_x\phi)^2+K(\partial_x\theta)^2\right]+\frac{g}{(2\pi a)^2}\cos\sqrt{16\pi}\,\phi$$

where

$$v_c^2 = v_F^2\left[1-\left(\frac{Va}{\pi v_F}\right)^2\cdot 4\right]$$

$$K = \sqrt{\frac{1-\frac{Va}{\pi v_F}\cdot 2}{1+\frac{Va}{\pi v_F}\cdot 2}} \quad\Rightarrow\quad K<1 \text{ if } V>0.$$

Next step, I will derive RG for the above sine-Gordon E.q. We Change it to Langrangean.

$$\mathcal{L} = \int dx\,d\tau\ \frac{1}{2vK}(\partial_t\phi)^2-\frac{v}{2K}(\partial_x\phi)^2-\frac{g}{(2\pi a)^2}\cos\sqrt{16\pi}\,\phi$$

imaginary time

$$\longrightarrow \int dx\,d\tau\ \frac{1}{2}\left[\frac{(\partial_\tau\phi')^2}{v}+v(\partial_x\phi')^2\right]+\frac{g}{(2\pi a)^2}\cos\beta\phi', \quad \text{where } \phi'=\frac{\phi}{\sqrt{K}}$$

$$\beta=\sqrt{16\pi K}$$

action e^{-S}, where $S = S_0+S_I = \frac{1}{2}\int dx\,d\tau\left[(\partial_\tau\phi')^2+(\partial_x\phi')^2\right]$

$$+\frac{g}{(2\pi a)^2}\cos\beta\phi' \quad (\text{set } v=1)$$

Hamiltonian

$$H_0 = v_F \int dx\, (\partial_x \phi_R)^2 + (\partial_x \phi_L)^2 \qquad \phi = \phi_R + \phi_L$$

$$= \frac{v_F}{2} \int dx\, (\partial_x \phi)^2 + (\partial_x \theta)^2 \qquad \theta = \phi_R - \phi_L$$

$$[\phi(x), \partial_{x'} \theta(x')] = -i\delta(x-x')$$

$$\partial_x \theta = -\Pi_\phi$$

add interaction - forward scattering

$$H = \frac{v}{2} \int dx\, \frac{1}{K}(\partial_x \phi)^2 + K(\partial_x \theta)^2 = \frac{v}{2} \int dx\, \frac{1}{K}(\partial_x \phi)^2 + K\,\Pi_\phi^2(x)$$

$\boxed{v \text{ and } K}$ are constant depends on interaction parameter

$$\rightarrow \mathcal{L} = \Pi_\phi \dot{\phi} - \mathcal{H} \qquad \dot{\phi} = \frac{\partial H}{\partial \Pi} = vK\,\Pi_\phi(x)$$

$$= \frac{\dot{\phi}^2}{vK} - \frac{v}{2K}(\partial_x \phi)^2 - \frac{vK}{2}\left(\frac{\dot{\phi}}{vK}\right)^2$$

$$= \frac{1}{2K}\left[\frac{1}{v}\left(\frac{\partial \phi}{\partial t}\right)^2 - v\left(\frac{\partial \phi}{\partial x}\right)^2\right]$$

$$L = \int dx \int dt\, \frac{1}{2K}\left[\frac{1}{v}(\partial_t \phi)^2 - v(\partial_x \phi)^2\right]$$

define
$$\begin{cases} \tilde{\Phi} = \frac{1}{\sqrt{K}}\phi = \frac{1}{\sqrt{K}}(\phi_R + \phi_L) \\ \tilde{\theta} = \sqrt{K}\,\theta = \sqrt{K}(\phi_R - \phi_L) \end{cases} \Rightarrow \boxed{\begin{aligned} \phi_R &= \frac{\sqrt{K}\tilde{\Phi}}{2} + \frac{\tilde{\theta}}{2\sqrt{K}} \\ \phi_L &= \frac{\sqrt{K}\tilde{\Phi}}{2} - \frac{\tilde{\theta}}{2\sqrt{K}} \end{aligned}}$$

under $\tilde{\Phi}$, $\tilde{\theta}$, we have

$$H = \frac{v}{2}\int dx\, (\partial_x \tilde{\Phi})^2 + (\partial_x \tilde{\theta})^2 \Rightarrow \mathcal{L} = \frac{1}{2}\left[\frac{1}{v}\left(\frac{\partial \tilde{\Phi}}{\partial t}\right)^2 - v\left(\frac{\partial \tilde{\Phi}}{\partial x}\right)^2\right]$$

for the interacting Hamiltonian, it's the vaccum for $\underbrace{\tilde{\phi} \text{ and } \tilde{\theta}}$. diagonal version

the original field ϕ_R, ϕ_L and $\tilde{\phi}_R$, $\tilde{\phi}_L$

$$\begin{pmatrix} \phi_R \\ \phi_L \end{pmatrix} = \begin{bmatrix} \frac{1+K}{2\sqrt{K}} & \frac{K-1}{2\sqrt{K}} \\ \frac{K-1}{2\sqrt{K}} & \frac{1+K}{2\sqrt{K}} \end{bmatrix} \begin{bmatrix} \tilde{\phi}_R \\ \tilde{\phi}_L \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \tilde{\phi}_R \\ \tilde{\phi}_L \end{bmatrix} = \begin{bmatrix} \frac{1+K}{2\sqrt{K}} & \frac{1-K}{2\sqrt{K}} \\ \frac{1-K}{2\sqrt{K}} & \frac{1+K}{2\sqrt{K}} \end{bmatrix} \begin{bmatrix} \phi_R \\ \phi_L \end{bmatrix}$$

$\tilde{\phi}_R$ and $\tilde{\phi}_L$ should also be expanded in term of new mode $\tilde{b}_q$ $\tilde{b}_q^\dagger$

$$\tilde{\phi}_R = \tilde{\varphi}_R(x) + \tilde{\varphi}_R^\dagger(x) + \frac{\sqrt{\pi}x}{L} \tilde{\hat{N}}_R$$

$$\tilde{\phi}_L = \tilde{\varphi}_L(x) + \tilde{\varphi}_L^\dagger(x) + \frac{\sqrt{\pi}x}{L} \tilde{\hat{N}}_L$$

with $\tilde{\varphi}_R = \sqrt{\frac{1}{4\pi}} \sum_{q>0} \frac{1}{\sqrt{n_q}} \tilde{b}_q e^{iqx - aq/2}$, $\tilde{\varphi}_R^\dagger = \sqrt{\frac{1}{4\pi}} \sum_{q>0} \frac{1}{\sqrt{n_q}} \tilde{b}_q^\dagger e^{-iqx-aq/2}$

$\tilde{\varphi}_L^\dagger = \sqrt{\frac{1}{4\pi}} \sum_{q>0} \frac{1}{\sqrt{n_q}} \tilde{b}_{-q}^\dagger e^{+iqx - aq/2}$, $\tilde{\varphi}_L = \sqrt{\frac{1}{4\pi}} \sum_{q>0} \frac{1}{\sqrt{n_q}} \tilde{b}_{-q} e^{-iqx - aq/2}$

where
$$\begin{bmatrix} \tilde{b}_q \\ \tilde{b}_{-q}^\dagger \end{bmatrix} = \begin{bmatrix} \frac{1+K}{2\sqrt{K}} & \frac{1-K}{2\sqrt{K}} \\ \frac{1-K}{2\sqrt{K}} & \frac{1+K}{2\sqrt{K}} \end{bmatrix} \begin{bmatrix} b_q \\ b_{-q}^\dagger \end{bmatrix}$$

$$\hookrightarrow \begin{bmatrix} ch\theta & -sh\theta \\ -sh\theta & ch\theta \end{bmatrix} \quad \text{where} \quad \begin{pmatrix} ch\theta = \frac{1+K}{2\sqrt{K}} \\ sh\theta = \frac{K-1}{2\sqrt{K}} \end{pmatrix}.$$

Correlation function:

The ground state is the vaccum of $\tilde{b}_q$ operators.

$$[\tilde{\varphi}_R(x), \tilde{\varphi}_R^\dagger(x')] = \frac{-1}{4\pi}\ln\left[\frac{2\pi}{L}(a - i(x-x'))\right]$$

$$[\tilde{\varphi}_L(x), \tilde{\varphi}_L^\dagger(x')] = \frac{-1}{4\pi}\ln\left[\frac{2\pi}{L}(a + i(x-x'))\right]$$

which are the same as $\varphi_R\, \varphi_R^\dagger,\ \varphi_L, \varphi_L^\dagger$ before.

$$\langle G|\tilde{\phi}_R(xt)\tilde{\phi}_R(0)|G\rangle = \langle G|\tilde{\varphi}_R(x,t)\tilde{\varphi}_R^\dagger(0)|G\rangle$$

$$= \langle G|[\tilde{\varphi}_R(x,t), \tilde{\varphi}_R^\dagger(0)]|G\rangle = \frac{-1}{4\pi}\ln\left[\frac{2\pi}{L}(a - i(x-vt))\right]$$

$$\langle G|\tilde{\phi}_L(xt)\tilde{\phi}_L(0)|G\rangle = \langle G|[\tilde{\varphi}_L(xt), \tilde{\varphi}_L^\dagger(0)]|G\rangle = \frac{-1}{4\pi}\ln\left[\frac{2\pi}{L}(a + i(x+vt))\right]$$

$$\langle G|e^{i\beta\tilde{\phi}_R(xt)}\, e^{-i\beta'\tilde{\phi}_R(0)}|G\rangle$$

use the identity $e^A e^B = :e^{A+B}:\ e^{\langle G|AB + \frac{A^2}{2} + \frac{B^2}{2}|G\rangle}$

$$e^{i\beta\tilde{\phi}_R(xt)}\, e^{-i\beta'\tilde{\phi}_R(0)} = :e^{i\beta\tilde{\phi}_R(xt) - i\beta'\tilde{\phi}_R(0)}:\ e^{\langle G|\beta\beta'[\tilde{\phi}_R(xt)\tilde{\phi}_R(0) - \tilde{\phi}_R^2(0)]|G\rangle} \cdot e^{-\frac{(\beta-\beta')^2}{2}\langle G|\tilde{\phi}_R^2(0)|G\rangle}$$

$$\langle G|\tilde{\phi}_R^2(0)|G\rangle = \frac{-1}{4\pi}\ln\frac{2\pi a}{L} \Rightarrow e^{-\frac{(\beta-\beta')^2}{2}\langle G|\tilde{\phi}_R^2(0)|G\rangle} = e^{-\frac{(\beta-\beta')^2}{8\pi}\ln\frac{L}{2\pi a}}$$

$$= \left(\frac{L}{2\pi a}\right)^{-\frac{(\beta-\beta')^2}{8\pi}} \xrightarrow[\text{if } \beta\neq\beta']{} 0$$

thus we set $\beta = \beta'$

$$e^{\beta^2\langle G|\tilde{\phi}_R(xt)\tilde{\phi}_R(0) - \tilde{\phi}_R^2(0)|G\rangle} = e^{\beta^2\left(\frac{-1}{4\pi}\right)\ln\frac{a - i(x-vt)}{a}}$$

$$= \left[\frac{a}{a - i(x-vt)}\right]^{\frac{\beta^2}{4\pi}}$$

$$\boxed{\langle G| e^{i\beta\tilde{\phi}_R(x t)}\, e^{-i\beta'\tilde{\phi}_R(0)}|G\rangle = \delta_{\beta\beta'}\left[\frac{a}{a-i(x-vt)}\right]^{\frac{\beta^2}{4\pi}}}$$

Similarly

$$\langle G| e^{i\beta\tilde{\phi}_L(xt)}\, e^{-i\beta'\tilde{\phi}_L(0)}|G\rangle = \delta_{\beta\beta'}\langle G|:e^{i\beta(\tilde{\phi}_L(xt)-\tilde{\phi}_L(0))}:\, e^{\beta^2\langle G|\tilde{\phi}_L(xt)\tilde{\phi}_L(0)-\tilde{\phi}_L^2(0)|G\rangle}|G\rangle$$

$$= \delta_{\beta\beta'}\, e^{\beta^2\langle G|\tilde{\phi}_L(xt)\tilde{\phi}_L(0)-\tilde{\phi}_L^2(0)|G\rangle} = \delta_{\beta\beta'}\, e^{\beta^2\left(\frac{-1}{4\pi}\right)\ln\frac{a+i(x+vt)}{a}}$$

$$\boxed{\langle G| e^{i\beta\tilde{\phi}_L(xt)}\, e^{-i\beta\tilde{\phi}_L(0)}|G\rangle = \delta_{\beta\beta'}\left[\frac{a}{a+i(x+vt)}\right]^{\frac{\beta^2}{4\pi}}}$$

Correlation function (II)

$$\langle G| e^{i\beta\phi(xt)}\, e^{-i\beta\phi(0)}|G\rangle = \langle G| e^{i\beta\sqrt{K}\tilde{\phi}(xt)}\, e^{-i\beta\sqrt{K}\tilde{\phi}(0)}|G\rangle$$

$$\cancel{= \langle G| e^{i\beta\sqrt{K}\tilde{\phi}_R(xt)}\, e^{i\beta\sqrt{K}\tilde{\phi}_{RL}(xt)}\, e^{\beta^2 K\frac{i}{4}}}$$

$$e^{i\beta\sqrt{K}\tilde{\phi}(xt)}\, e^{-i\beta\sqrt{K}\tilde{\phi}(0)} = :e^{-i\beta\sqrt{K}(\tilde{\phi}(xt)-\tilde{\phi}(0))}:\, e^{\langle G|\beta^2K^2(\tilde{\phi}(xt)\tilde{\phi}(0)-\tilde{\phi}^2(0))|G\rangle}$$

$$\langle G|\tilde{\phi}(xt)\tilde{\phi}(0)-\tilde{\phi}^2(0)|G\rangle = \langle G|(\tilde{\phi}_R(xt)+\tilde{\phi}_L(xt))(\tilde{\phi}_R(0)+\phi_L(0)) - (\tilde{\phi}_R(0)+\tilde{\phi}_L(0))^2|G\rangle$$

$$= \langle G|\tilde{\phi}_R(xt)\tilde{\phi}_R(0)-\tilde{\phi}_R^2(0)|G\rangle + \langle G|\tilde{\phi}_L(xt)\tilde{\phi}_L(0)-\tilde{\phi}_L^2(0)|G\rangle$$

$$\Rightarrow \langle G| e^{i\beta\phi(xt)}\, e^{-i\beta\phi(0)}|G\rangle = \left[\frac{a}{a-i(x-vt)}\right]^{\frac{\beta^2K}{4\pi}}\left[\frac{a}{a+i(x+vt)}\right]^{\frac{\beta^2K}{4\pi}}$$

$$\boxed{\langle G|\, e^{i\beta\phi(x,t)}\, e^{-i\beta\phi(0)}\,|G\rangle = \left[\frac{a^2}{(a+ivt)^2+x^2}\right]^{\frac{\beta^2 K}{4\pi}}}$$

$\Rightarrow$ Scaling dimension of $O = e^{i\beta\phi}$, $\Delta = \frac{\beta^2 K}{4\pi}$

For example, for CDW/BW order $N^{\dagger} = \frac{i}{2\pi a} e^{i\sqrt{4\pi}\phi} \Rightarrow \boxed{\Delta_N = K.}$

$$\langle G|e^{i\beta\theta(x,t)}\, e^{-i\beta\theta(0)}|G\rangle = \langle G|\, e^{i\beta\frac{\tilde{\theta}(x,t)}{\sqrt{K}}}\; e^{-i\beta\frac{\tilde{\theta}(0)}{\sqrt{K}}}\,|G\rangle$$

$$e^{i\frac{\beta}{\sqrt{K}}\tilde{\theta}(x,t)}\, e^{-i\frac{\beta}{\sqrt{K}}\tilde{\theta}(0)} = :e^{-i\frac{\beta}{\sqrt{K}}\tilde{\theta}(x,t)-\tilde{\theta}(0)}:\; e^{\langle G|\frac{\beta^2}{K^2}[\tilde{\theta}(x,t)\tilde{\theta}(0)-\tilde{\theta}^2(0)]|G\rangle}$$

$$\langle G|\tilde{\theta}(x,t)\tilde{\theta}(0)-\tilde{\theta}^2(0)|G\rangle = \langle G|\tilde{\phi}_R(x,t)\phi_R(0)-\tilde{\phi}_R^2(0)|G\rangle\langle G|\tilde{\phi}_L(x,t)\phi_L(0)-\tilde{\phi}_L^2(0)|G\rangle$$

$$\tilde{\theta} = \tilde{\phi}_R - \tilde{\phi}_L$$

$$\Rightarrow \boxed{\langle G|\, e^{i\beta\theta(x,t)}\, e^{-i\beta\theta(0)}\,|G\rangle = \left[\frac{a^2}{(a+ivt)^2+x^2}\right]^{\frac{\beta^2}{4\pi K}}}$$

Scaling dimension of $O = e^{i\beta\theta} \Rightarrow \Delta = \frac{\beta^2}{4\pi}K^{-1}$

for pairing $\Delta_{pair} = \frac{i}{2\pi a} e^{-i\sqrt{4\pi}\theta} \Rightarrow \boxed{\Delta_{\Delta_{pair}} = K^{-1}}$

attractive

Thus in the Luttinger liquid phase, there are two competing ordering CDW with scaling dimension K, pairing . . . . K^{-1}.

$\boxed{\text{If } K>1, \text{ pairing wins} \quad K<1 \text{ CDW wins}}$

repulsive

Correlation function (III) - fermion

$$\psi_R = \frac{1}{\sqrt{2\pi a}}\, e^{i\sqrt{4\pi}\,\phi_R}$$

$$\langle G|\,\psi_R(xt)\,\psi_R^\dagger(0)\,|G\rangle = \frac{1}{2\pi a}\langle G|\,\underbrace{e^{i\sqrt{4\pi}\,\phi_R(xt)}\,e^{-i\sqrt{4\pi}\,\phi_R(0)}}_{:e^{i\sqrt{4\pi}(\phi_R-\phi_R(0))}:\; e^{4\pi\langle G|\phi_R(xt)\phi_R(0)-\phi_R^2(0)|G\rangle}}\,|G\rangle$$

$$\langle G|\,\phi_R(xt)\phi_R(0)-\phi_R^2(0)\,|G\rangle = \langle G|\left(\frac{1+K}{2\sqrt K}\tilde\phi_R + \frac{1-K}{2\sqrt K}\tilde\phi_L\right)\left(\frac{1+K}{2\sqrt K}\tilde\phi_R(0)+\frac{1-K}{2\sqrt K}\tilde\phi_L(0)\right) - \cdots|G\rangle$$

$$= \frac{(1+K)^2}{4K}\langle G|\,\tilde\phi_R(xt)\,\tilde\phi_R(0)-\tilde\phi_R^2(0)\,|G\rangle + \frac{(1-K)^2}{4K}\langle G|\,\tilde\phi_L(xt)\,\tilde\phi_L(0)-\tilde\phi_L^2(0)\,|G\rangle$$

$$\boxed{\langle G|\,\psi_R(xt)\,\psi_R^\dagger(0)\,|G\rangle} = \exp\left[-\frac{(1+K)^2}{4K}\ln\frac{a-i(x-vt)}{a} - \frac{(1-K)^2}{4K}\ln\frac{a+i(x-vt)}{a}\right]$$

$$\boxed{= \frac{1}{2\pi a}\left[\frac{a}{a-i(x-vt)}\right]^{\frac{(1+K)^2}{4K}}\left[\frac{a}{a+i(x+vt)}\right]^{\frac{(1-K)^2}{4K}}}$$

$$\psi_L = \frac{1}{\sqrt{2\pi a}}\, e^{-i\sqrt{4\pi}\,\phi_L}$$

$$\langle G|\,\psi_L(xt)\,\psi_L^\dagger(0)\,|G\rangle = \frac{1}{2\pi a}\langle G|\,e^{-i\sqrt{4\pi}\,\phi_L(xt)}\,e^{i\sqrt{4\pi}\,\phi_L(0)}\,|G\rangle$$

$$\langle G|\,\phi_L(xt)\phi_L(0)-\phi_L^2(0)\,|G\rangle = \frac{(1-K)^2}{4K}\langle G|\,\tilde\phi_R(xt)\,\phi_R(0)-\phi_R^2(0)\,|G\rangle$$
$$+\frac{(1+K)^2}{4K}\langle G|\,\tilde\phi_L(xt)\,\phi_L(0)-\phi_L^2(0)\,|G\rangle$$

$$\boxed{\Rightarrow \langle G|\,\psi_L(xt)\,\psi_L^\dagger(0)\,|G\rangle = \frac{1}{2\pi a}\left[\frac{a}{a-i(x-vt)}\right]^{\frac{(1+K)^2}{4K}}\left[\frac{a}{a+i(x+vt)}\right]^{\frac{(1-K)^2}{4K}}}$$

fermion scaling dimension larger than 1/2.

$$\boxed{\Delta_\psi = \frac{1}{2}\left[\frac{(1+K)^2}{4K} + \frac{(1-K)^2}{4K}\right] = \frac{1}{4}(K + 1/K) \geq \frac{1}{2}}$$

how about density / current correlation?

$$\langle G | \partial_x \phi(x t)\; \partial_x \phi(0 0) | G\rangle = \partial_{x_1}\partial_{x_2}\langle G | \phi(x_1 t_1)\, \phi(x_2 t_2) - \phi^2(0) | G\rangle$$

$$= K\; \partial_{x_1}\partial_{x_2} \langle G | \tilde\phi(x_1 t_1)\tilde\phi(x_2 t_2) - \tilde\phi^2(0) | G\rangle$$

$$= K\partial_{x_1}\partial_{x_2}\{\langle G|\tilde\phi_R(x_1t_1)\tilde\phi_R(x_2t_2) - \tilde\phi_R(0)\,|G\rangle + R\to L\rangle$$

$$= K\,\partial_{x_1}\partial_{x_2}\left[\left(\frac{-1}{4\pi}\right)\ln[a+i(x+vt)] + \left(\frac{-1}{4\pi}\right)\ln(a-i(x-vt))\right]$$

$$= \frac{K}{4\pi}\left[\partial_x^2 \ln a+i(x+vt) + \partial_x^2 \ln(a-i(x-vt))\right]$$

$$= +\frac{K}{4\pi}\left[\frac{1}{[a+i(x+vt)]^2} + \frac{1}{[a-i(x-vt)]^2}\right]$$

it's negatively correlated!