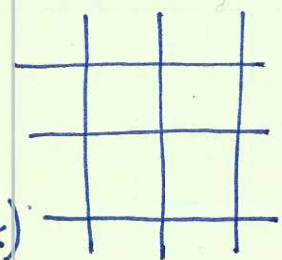
Digression — high temperature, low temperature expansion, duality

Consider Ising model (2D), $Z = \sum_{\{\sigma_i\}} \exp[K \sum_{\langle ij \rangle} \sigma_i \sigma_j]$.



☆ Starting from the limit $K \to 0$, (high T expansion)

$$e^{K\sigma_i\sigma_j} = \cosh K + \sigma_i\sigma_j \sinh K = \cosh K (1 + \sigma_i\sigma_j \tanh K)$$

$$\Rightarrow Z = \sum_{\{\sigma_i\}} \prod_{\text{bond } ij} \cosh K (1 + \sigma_i \sigma_j \tanh K)$$

There are 2^{2N} bond $\to (\cosh K)^{2N}$, and $\sum_{\{\sigma_i\}}$ sum over 2^N configurations

Hence the leading term $2^N (\cosh K)^{2N}$. We will expand

$$\frac{Z(K)}{2^N (\cos K)^{2N}} = 1 + \text{high orders of } \tanh K + \cdots$$

☆ consider order of $\tanh K$,

it needs a bond ij and 1 from other place, but $\sum_{\sigma_i \sigma_j} \sigma_i \sigma_j$

tanhK from

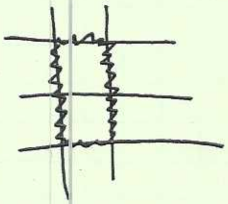
gives to zero. The lesson here is that we cannot have free dangling spins. The bonds for tanhK must form loops.

Then the smallest loop contains 4 sites. — 

Then each spin appears twice, ~~since each σ_i~~ then $\sigma_i^2 = 1$. $\Rightarrow$

$$\underset{\substack{\swarrow \\ \text{sum over } \{\sigma_i\}}}{2^N} \cdot \underset{\substack{\downarrow \\ 2^N \text{ bond}}}{(\cos K)^{2N}} \cdot \underbrace{N \tanh^4 K}_{N\text{-plaquette}}$$

Then the next order loops are 6- lengths

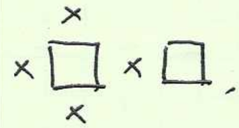


Since the rectangle can have 2 different directions.

we have $2N \tanh^6 K$.

$$\Rightarrow \quad \frac{Z(K)}{2^N (\cosh K)^{2N}} = 1 + N \tanh^4 K + 2N \tanh^6 K + \cdots\cdots \frac{N(N+9)}{2} \tanh^8 K$$

The order of $\tanh^8 K$ is more trickly: there are $\frac{N(N-5)}{2}$ configures of dis connected loops of two squares:



and there're also $7N$ configurations of single loops.

$\rightarrow \frac{N(N+9)}{2}$

Hence: we have
$$\frac{Z(K)}{2^N (\cos K)^{2N}} = \sum_{\substack{\text{closed} \\ \text{loops}}} C(L) \tanh^L K$$
$C(L)$: # of loops

⊛ Low-T expansion: $K \to \infty$

There are $2N$-bond, Hence the leading one $e^{2NK} \cdot 2$ (fully spin up or fully spin down)

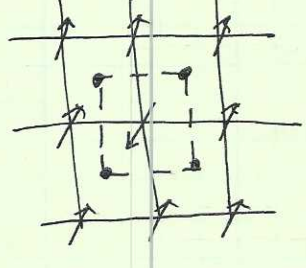
Then flip one spin, $\rightarrow$ break 4 bonds., energy cost $8K$

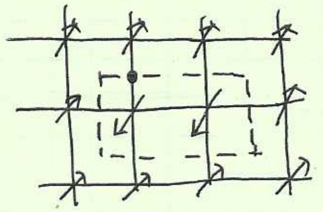
flip two adjacent spins $\longrightarrow$ break 6 bonds, energy cost $16K$

$$\Rightarrow \quad \frac{Z}{2 \cdot e^{2NK}} = (1 + N e^{-8K} + 2N e^{-12K} + \cdots \quad)$$

Actually, we can find a correspondence. If we draw the bisecters of the broken bonds, we also form closed loops.

In other words, its the loop on the dual lattice.





$\Rightarrow$ There's a "one to one" correspondence of the loops of the high T expansion, and low T expansions. The loop in the low T case in just the domain walls.

Hence if we set $\boxed{\tanh K = e^{-2K^*}}$ $\longleftrightarrow$ $\sinh 2K \sinh 2K^* = 1$ hence $(K^*)^* = K$.

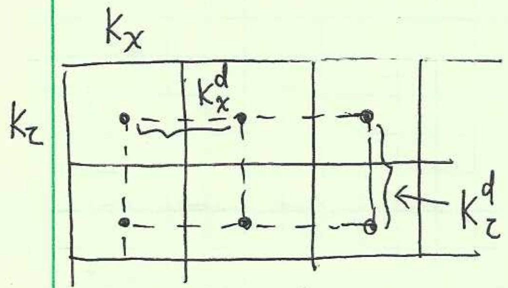
we have

$$\boxed{\frac{Z(K)}{2^N(\cosh K)^{2N}} = \frac{Z(K^*)}{2 \cdot e^{2NK^*}}}$$

$\leftarrow$ mapping an Ising model with K to another one with K^*.

The phase transition, i.e. the singular point of $\ln Z(K)$, if there's only one, it must occure at $K = K^*$, i.e $\sinh 2K_c = 1 \Rightarrow K_c = \frac{1}{2}[\ln(\sqrt{2}+1)]$

(*) Duality to the anisotropic Ising model (2D)



Let's repeat the above analysis, since the bond of the dual lattice (low T expansion) ~~in the~~ intersect bonds with the different orientation. We need the correspondence

$$\tan K_x = e^{-2K_z^d}, \quad \tan K_z = e^{-2K_x^d}$$

$$\Rightarrow \begin{cases} K_\tau^d = K_x^* \\ K_x^d = K_\tau^* \end{cases}$$ and X^* is still defined as $\sinh 2X \sinh 2X^* = 1$

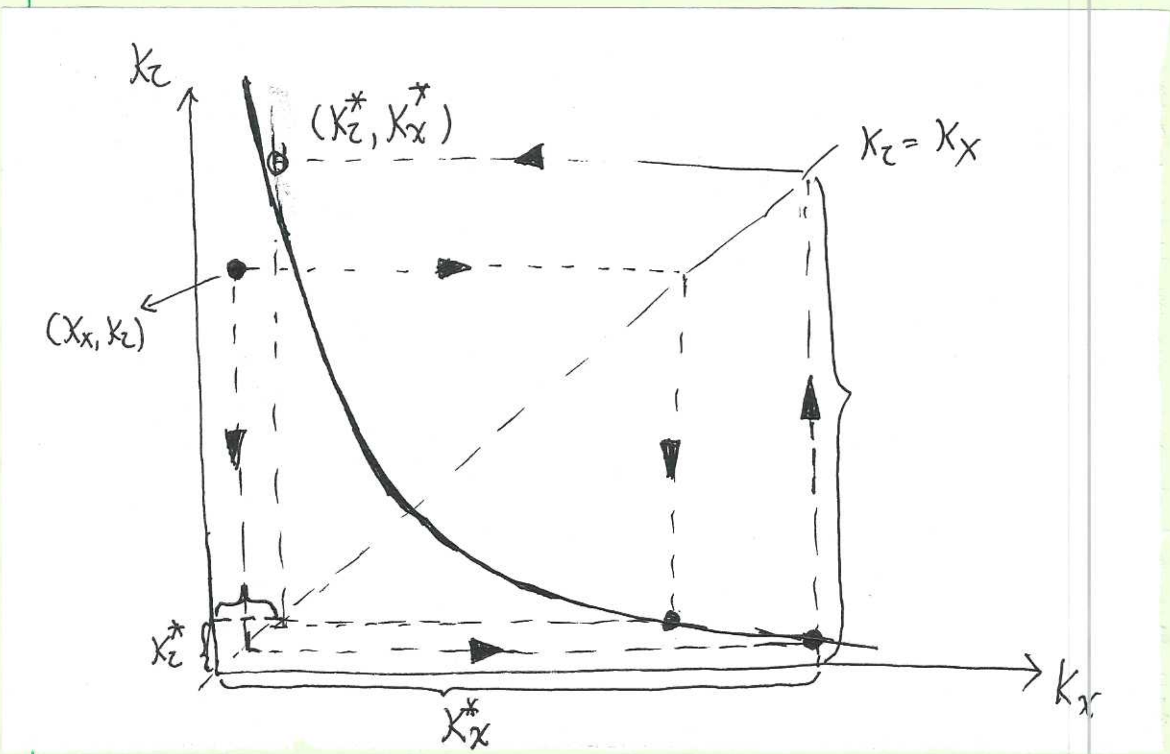
i.e. $(K_x, K_\tau) \xrightarrow{\text{dual}} (K_\tau^*, K_x^*)$.

The self-dual point, i.e. the critical points satisfies

$$\underline{K_x = K_\tau^*}, \Leftrightarrow X_x^* = (K_\tau^*)^* = K_\tau.$$

we have $\sinh 2K_x \sinh 2K_x^* = 1 \Leftrightarrow \boxed{\sinh 2K_x \sinh 2K_\tau = 1}$

Critical line of anisotropic Ising model.



2D Ising model, quantum 1D Ising chain, Majorana fermion

① Transfer matrix formalism – 1D Ising model

$$Z = \sum_{\{\sigma\}} e^{\beta J \sum_i (\sigma_i \sigma_{i+1} - 1)} = \sum_{\{\sigma_1, \cdots \sigma_n\}} T_{\sigma_1 \sigma_2} \cdots T_{\sigma_N \sigma_1} = tr\, T^N$$

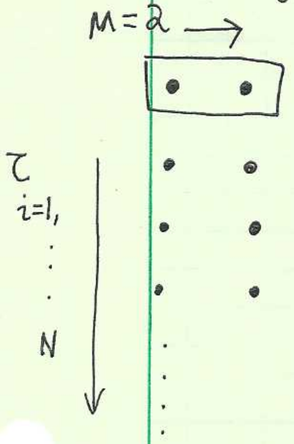
where $T_{\sigma_1\sigma_2} = e^{\beta J(\sigma_1 \sigma_2 - 1)} \Rightarrow T = \begin{pmatrix} 1 & e^{-2\beta J} \\ e^{-2\beta J} & 1 \end{pmatrix} = e^{h \Delta\tau \sigma_1}$

with the relation $\boxed{\sinh 2\beta J \sinh 2h\Delta\tau = 1}$

hence $Z = \sum_{\{\sigma\}} e^{\beta J \sum_i (\sigma_i \sigma_{i+1} - 1)} = tr\left[e^{N\Delta\tau h \sigma_1}\right] = tr\left[e^{-\beta_\tau h \sigma_1}\right]$

with $\beta_\tau = N\Delta\tau \xrightarrow{N\to\infty} \infty$

② Transfer matrix to 2- chain



$$Z = \sum_{\{\sigma\}} e^{\beta J \sum_{i=1}^{N} \sum_{j=1}^{2} \sigma(i,j)\sigma(i,j+1) + \sigma(i,j)\sigma(i+1,j))}$$

~~transfer~~ matrix is 4-dimensional

defin $S_i = (\sigma(i,1), \sigma(i,2))$ — taking 4 possible values

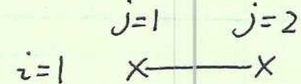
i.e. $S_i = (11)\,(-1+1)\,(+1,-1),\,(-1,-1)$, then

$$Z = \sum_{S_1, S_2 \ldots} T_{S_1 S_2} T_{S_2 S_3} \cdots T_{S_N S_1} = tr\, T^N$$

$$T_{S_1S_2} = e^{\beta J \sigma(1,1)\sigma(1,2)} \quad e^{\beta J \sum_{j=1}^{2} \sigma(1,j)\sigma(2,j)}$$

$$= (V_2)_{S_1S_1}(V_1')_{S_1S_2}$$

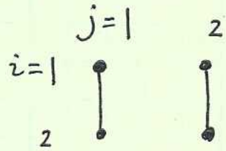
where V_2 is diagonal, describing the horizontal bond



$$V_2 = \left[e^{\beta J \overset{j=1}{\sigma_z(1)} \overset{j=2}{\sigma_z(2)}} \right]_{S_1S_1}$$

here $\sigma_z(1)\,\sigma_z(2)$ are quantum operators now.

V_1' describe the vertical bonds — independent evolution of two spins.



$$\begin{array}{c|cccc} S_1 \backslash S_2 & (1\ 1) & (-1,1) & (1\ -1) & (-1\ -1) \\ \hline (1\ 1) & & & & \\ (-1\ 1) & & \cdots & & \\ (1\ -1) & & & & \\ (-1,-1) & & & & \end{array} = \left[e^{h\Delta\tau\,\sigma_x(j=1)} \otimes e^{h\Delta\tau\,\sigma_x(j=2)} \right]_{S_1S_2}$$

$V_1' = a \otimes a$, and $\langle s|a|s'\rangle = e^{\beta s s'} = \begin{bmatrix} e^{\beta J} & e^{-\beta J} \\ e^{-\beta J} & e^{\beta J} \end{bmatrix} = e^{\beta J} + e^{-\beta J}\sigma_x$

introduce $\theta = h\Delta\tau$, such ~~tha~~ that $\tanh\theta = e^{-\beta J}/e^{\beta J} = e^{-2\beta J}$

we have $a = \sqrt{(e^{\beta J})^2 - (e^{-\beta J})^2}\; e^{\theta\sigma_x} = \sqrt{2\sinh 2\beta J}\; e^{\theta\sigma_x}$

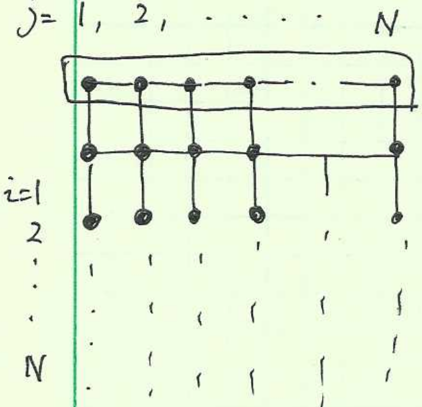
$$\Rightarrow \quad V_1' = \left[2\sinh 2\beta J\right]^{2/2} \; e^{\theta[\sigma_x(1) + \sigma_x(2)]}$$

we often denote $V_1 = e^{\theta[\sigma_x(1)+\sigma_x(2)]}$

$$\Rightarrow \quad \boxed{T = \left[2\sinh 2\beta J\right]^{2/2} \; V_2 \, V_1} \leftarrow \boxed{\begin{array}{l} V_2 = e^{\beta J\, \sigma_z(1)\,\sigma_z(2)} \\ \sinh 2\beta J \, \sinh 2\theta = 1 \end{array}}$$

$$\tanh\theta = e^{-2\beta J}$$

③ Now we can generalize the system to N-chain



define $S_i = (\sigma(i,1), \sigma(i,2), \cdots \sigma(i,N))$

↙ the configuration of the i-th row

$E(S_i, S_{i+1})$ represents the coupling between the i and $i+1$ th rows.

$E(S_i)$ represents the intra-i th chain coupling

$$E(S_i) = -J \sum_{j=1}^{N} \underbrace{\sigma(i,j)\,\sigma(i,j+1)} \qquad E(S_i, S_{i+1}) = -J \sum_{j=1}^{N} \sigma(i,j)\,\sigma(i+1,j)$$

$$\Rightarrow Z = \sum_{S_1} \cdots \sum_{S_n} \exp\left[-\beta\left\{\sum_{i=1}^{n} E(S_i, S_{i+1}) + E(S_i)\right\}\right]$$

$$\text{define } \langle s|T|s'\rangle \equiv e^{-\beta(E(s,s') + E(s))}$$

$$T_{ss'} = \underbrace{(V_2)_{ss}}_{[2\sinh 2\beta J]^{N/2}} (V_1)_{ss'}, \quad \text{with } V_2 = e^{\beta J \sum_{j=1}^{N} \sigma_z(j)\sigma_z(j+1)}$$

$$V_1 = e^{\theta \sum_{j=1}^{N} \sigma_x(j)}$$

and $\sinh 2\beta J \sinh 2\theta = 1$

and $Z = tr[T^N]$

where T is an $2^N \times 2^N$ dimensional matrix.

(4) Majorana fermion representation —— Jordan Wigner transformat

a square lattice $N \times N$, $\rightarrow$ Classic Ising model: summing over 2^{N^2} configurations

$\rightarrow$ map to Quantum transfer matrix T: a problem in $2^N \times 2^N$ matrix.

i.e. Hilbert space of 2^N dimensional.

Hence, we map a problem of 2^{N^2} classic configurations

$\rightarrow$ 2^N dimensional Hilbert space.

The next step is to reorganize it into this Hilbert space of spin (hard core bosons) into free fermion's.

The method is the Jordan-Wigner transformation define $\psi_1(n) = \begin{cases} \sigma_y(1) & \text{for } n=1 \\ \left(\prod_{l=1}^{n-1} \sigma_x(l)\right) \sigma_y(n) & \text{for } n>1 \end{cases}$

$$\psi_2(n) = \begin{cases} \sigma_z(1) & \text{for } n=1 \\ \left(\prod_{l=1}^{n-1} \sigma_x(l)\right) \sigma_z(n) & \text{for } n>1 \end{cases}$$

$$\rightarrow \quad \begin{array}{c|c|c|} \begin{matrix}\sigma_y\\ \sigma_z\end{matrix} & 1 & 1 \\ \hline \begin{matrix}\sigma_x\\ \sigma_x\end{matrix} & \begin{matrix}\sigma_y\\ \sigma_z\end{matrix} & 1 \\ \hline \begin{matrix}\sigma_x\\ \sigma_x\end{matrix} & \begin{matrix}\sigma_x\\ \sigma_x\end{matrix} & \begin{matrix}\sigma_y\\ \sigma_z\end{matrix} \\ \hline \end{array} \quad \longrightarrow \quad \begin{array}{c|c|c|c} \begin{matrix}\sigma_y\\ \sigma_z\end{matrix} & 1 & 1 & 1 \\ \hline \begin{matrix}\sigma_x\\ \sigma_x\end{matrix} & \begin{matrix}\sigma_y\\ \sigma_z\end{matrix} & 1 & 1 \\ \hline \begin{matrix}\sigma_x\\ \sigma_x\end{matrix} & \begin{matrix}\sigma_x\\ \sigma_x\end{matrix} & \begin{matrix}\sigma_y\\ \sigma_z\end{matrix} & 1 \\ \hline \begin{matrix}\sigma_x\\ \sigma_x\end{matrix} & \begin{matrix}\sigma_x\\ \sigma_x\end{matrix} & \begin{matrix}\sigma_x\\ \sigma_x\end{matrix} & \begin{matrix}\sigma_y\\ \sigma_z\end{matrix} \\ \hline \end{array}$$

it's easy to check that all the $\psi_1(n)$ and $\psi_2(n)$ anti commute

and we can also define $\Gamma_5 = \sigma_x \otimes \sigma_x \otimes \cdots \sigma_x$. which anti commute with all $\psi_1(n)$ and $\psi_2(n)$, for $n = 1, 2, \cdots N$.

Then
$$\sigma_x(n) = -i\,\psi_1(n)\,\psi_2(n)$$
$$\sigma_z(n)\,\sigma_z(n+1) = i\,\psi_1(n)\,\psi_2(n+1)$$

but each $\sigma_z(n) = \prod_{l=1}^{n-1} \left(-i\,\psi_1(l)\,\psi_2(l)\right) \psi_2(n)$

Hence, the Ising model in the longitudinal field, which couples to $\sigma_z(n)$, remains a difficult problem.

Now we can formulate

$$T = \exp\left[\beta J \sum_{j=1}^{N} \sigma_z(j)\,\sigma_z(j+1)\right] \exp\left[\theta \sum_{j=1}^{N} \sigma_x(j)\right]$$

$$= e^{i\beta J \sum_{j=1}^{N} \psi_1(j)\psi_2(j+1)} \; e^{-i\theta \sum_{j=1}^{N} \psi_1(j)\psi_2(j)}$$

From now on, we use $K_x = \beta J$, $K_z^* = \theta$, $*$ mean the dual expression

since $\sin 2K_x \sinh 2\theta = 1$ or

Then $T = e^{iK_x \sum_{j=1}^{N} \psi_1(j)\psi_2(j+1)} \; e^{-iK_z^* \sum_{j=1}^{N} \psi_1(j)\psi_2(j)}$ $\quad \tanh\theta = e^{-2\beta J}$

㊟ Boundary condition

We want periodical boundary condition for the spins, i.e.

the last term in $\sum_{j=1}^{N} \sigma_z(j)\sigma_z(j+1)$, $\rightarrow \sigma_z(N)\,\sigma_z(1)$, or $\sigma_z(N+1) = \sigma_z(1)$

~~$\sigma_z(N)\,\sigma_z(1)$ ... $\psi_1(N)\psi_2(1)$~~

$$\psi_1(N)\psi_2(1) = \prod_{l=1}^{N-1} \sigma_x(l)\;\sigma_y(N)\;\sigma_z(1) = \prod_{l=1}^{N} \sigma_x(l)(+i)\,\sigma_z(N)\,\sigma_z(1)$$

$$\Rightarrow \quad \sigma_z(N)\,\sigma_z(1) = -i \prod_{l=1}^{N} \sigma_x(l)\;\psi_1(N)\psi_2(1)$$

$$= -i\,\Gamma_5\,\psi_1(N)\psi_2(1)$$

Compare $i\sum_{j=1}^{N-1} \psi_1(j)\psi_2(j+1)$, we have an extra $-\Gamma_5$ factor.

Forturnately, $\Gamma_5 = \prod_{l=1}^{N} \sigma_x(l)$ anticommute with all $\psi_{1,2}(j)$, hence

Γ_5 commutes with all the bilinear terms. We can discuss within

the eigensector of $\Gamma_5 = \pm 1$. Hence in the even/odd sectors of Γ_5, ~~it correspond~~

the periodical boundary condition of σ_z, corresponds to the

$$\boxed{\begin{array}{l} \text{periodical boundary condition } \psi_2(N+1) = -\psi_2(1) \leftarrow \Gamma_5 = -1, \\ \text{and anti-periodical boundary condition } \psi_2(N+1) = \psi_2(1) \leftarrow \Gamma_5 = 1. \end{array}}$$

⊛ Physical meaning of Γ_5

defin $\Psi(j) = \frac{1}{2}(\psi_1(j) + i\psi_2(j))$, $\Psi^\dagger(j) = \frac{1}{2}(\psi_1(j) - i\psi_2(j))$

Then it's easy to check $\{\Psi(j)\ \Psi^\dagger(i)\} = \delta_{ij}$, ← the fermion commutation relation.

Then $\Psi^\dagger(j)\Psi(j) = \frac{1}{2}(1 + i\psi_1(j)\psi_2(j)) = \frac{1}{2}(1 - \sigma_1(j))$

$$\Rightarrow \Gamma_5 = \sigma_1(1) \otimes \sigma_1(2) \cdots \otimes \sigma_1(N) = \prod_{j=1}^{N} (1 - 2\Psi^\dagger(j)\Psi(j)) = \prod_{j=1}^{N} (-)^{N_j} = (-)^{N_\Psi}$$

with $N_\Psi = \sum_{j=1}^{N} \Psi^\dagger(j)\Psi(j)$. Hence Γ_5 is the fermion parity operator.

For ψ_2, anti-periodical boundary for the sector of even fermion number
periodical boundary for the sector of odd fermion number.

We will assume N is a multiple of 4, then in the Fourier transform

$$k = \frac{m\pi}{N} \quad \text{with} \quad m = \pm 1, \pm 3, \cdots \pm(N-1)$$

for anti-periodical BC.

with $m = 0, \pm 2, \pm 4, \pm \cdots \pm(N-2), N$

for periodical BC.

⊗ Fermion number even sector

the normalization $\sqrt{N/2}$ is consist. with $\psi^2=1$

$$\psi_\alpha(j) = \frac{1}{\sqrt{N/2}} \sum_{m=-(N-1)}^{N-1} C_\alpha(k_m)\, e^{ik_m j} \quad (m: \text{odd}),\ \alpha=1,2$$

since $\psi_\alpha(j)$ is Majorana, $\Rightarrow$ $C_\alpha(k_m) = C^\dagger_\alpha(-k_m)$, satisfying $\{C, C^\dagger\}=1$.

$$\Rightarrow \psi_\alpha(j) = \frac{1}{\sqrt{N/2}} \sum_{m=1,3,\cdots N-1} \left[C_\alpha(k_m)\, e^{ik_m j} + C^\dagger_\alpha(k_m)\, e^{-ik_m j} \right]$$

Then

$$\sum_{j=1}^{N} \psi_1(j)\psi_2(j+1) = \frac{1}{(N/2)} \sum_{m,m'} \sum_{j=1}^{N} \left(C_1(k_m)\, e^{ik_m j} + C^\dagger_1(k_m)\, e^{-ik_m j} \right) \left(C_2(k_{m'})\, e^{ik_{m'}(j+1)} + C^\dagger_2(k_{m'})\, e^{-ik_{m'}(j+1)} \right)$$

$$= 2\sum_m \left[C_1(k_m)\, C^\dagger_2(k_m)\, e^{-ik_m} + C^\dagger_1(k_m)\, C_2(k_m)\, e^{ik_m} \right]$$

$$\sum_{j=1}^{N} \psi_1(j)\psi_2(j) = 2\sum_m C_1(k_m)\, C^\dagger_2(k_m) + C^\dagger_1(k_m)\, C_2(k_m)$$

$$\Rightarrow T_E = \exp\left[i2K_x \sum_m C_1(k_m) C^\dagger_2(k_m)\, e^{-ik_m} + C^\dagger_1(k_m)\, C_2(k_m)\, e^{ik_m} \right]$$

$$\cdot \exp\left[-i2K^*_\tau \sum_m C_1(k_m) C^\dagger_2(k_m) + C^\dagger_1(k_m)\, C_2(k_m) \right]$$

$$= \prod_{\otimes\, m=1,3,\cdots}^{N-1} T(m), \text{ where } \boxed{T(m) = e^{2iK_x\left[C_1(k_m) C^\dagger_2(k_m)\, e^{-ik_m} + C^\dagger_1(k_m) C_2(k_m)\, e^{ik_m} \right]} \cdot e^{-2iK^*_\tau \left[C_1(k_m) C^\dagger_2(k_m) + C^\dagger_1(k_m) C_2(k_m) \right]}}$$

$C_1(k_m)$, $C^\dagger_1(k_m)$, $C_2(k_m)$, $C^\dagger_2(k_m)$ live in 4-dim space. In terms of $C^\dagger_1 C_1$ and $C^\dagger_2 C_2$, these states are $|00\rangle$, $|01\rangle$ and $|10\rangle$, $|11\rangle$. Actually $|00\rangle$ and $|11\rangle$ can be projected out, since $C_1 C^\dagger_2 |00\rangle = C_1 C^\dagger_2 |11\rangle = 0$.

⑨

$T(m)$ in $|00\rangle$ and $|11\rangle$ will be just an identity operater

we only need to diagonalize $T(m)$ in the 2-dim sub-space $|01\rangle$ and $|10\rangle$

⊗ Fermion # odd sector

$$\begin{cases} C_\alpha(0) = C_\alpha^\dagger(0) = \eta_\alpha(0), \\ C_\alpha(\pi) = C_\alpha^\dagger(\pi) = \eta_\alpha(\pi) \end{cases} \Rightarrow \psi_\alpha(j) = \frac{1}{\sqrt{N/2}} \sum_{m=2,4\ldots}^{N-2} \Big\{ \left[C_\alpha(k_m) e^{ik_m j} + C_\alpha^\dagger(k_m) e^{-ik_m j} \right] + \eta_\alpha(0) + \eta_\alpha(\pi)(-)^j \Big\}$$

$$\Rightarrow \sum_{j=1}^{N} \psi_1(j)\psi_2(j+1) = \frac{1}{\left(\frac{N}{2}\right)} \sum_{mm'} \sum_j \left(C_1(k_m) e^{ik_m j} + C_1^\dagger(k_m) e^{-ik_m j} + \eta_1(0) + \eta_1(\pi)(-)^j \right) \left(C_2(k_{m'}) e^{ik_{m'}(j+1)} + C_2^\dagger(k_{m'}) e^{-ik_{m'}(j+1)} + \eta_2(0) + \eta_2(\pi)(-)^{j+1} \right)$$

$$= 2 \sum_{m=2,4\ldots}^{N-2} C_1(k_m) C_2^\dagger(k_m) e^{-ik_m} + C_1^\dagger(k_m) C_2(k_m) e^{ik_m} + \eta_1(0)\eta_2(0) - \eta_1(\pi)\eta_2(\pi)]$$

$$\sum_{j=1}^{N} \psi_1(j)\psi_2(j) = 2 \sum_{m=2,4\ldots}^{N-2} \left[C_1(k_m) C_2^\dagger(k_m) + C_1^\dagger(k_m) C_2(k_m) + \eta_1(0)\eta_2(0) + \eta_1(\pi)\eta_2(\pi) \right]$$

$$\Rightarrow T_{odd} = \exp\left[i2K_x \sum_{m=2,4\ldots}^{N-2} C_1(k_m) C_2^\dagger(k_m) e^{-ik_m} + C_1^\dagger(k_m) C_2(k_m) e^{ik_m} \right]$$

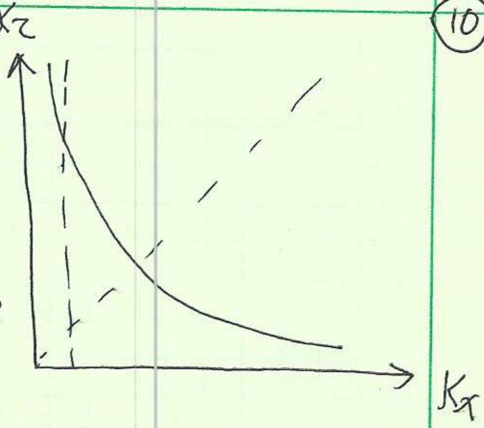
$$\cdot \exp\left[-i2K_z^* \sum_{m=2,4\ldots}^{N-2} C_1(k_m) C_2^\dagger(k_m) + C_1^\dagger(k_m) C_\alpha(k_m) \right]$$

$$\cdot \exp\left[(i2K_x - i2K_z^*) \eta_1(0)\eta_2(0) \right] \cdot \exp\left[-i2(K_x + K_z^*) \eta_1(\pi)\eta_2(\pi) \right]$$

$$\Rightarrow \boxed{ T_{odd} = \prod_{\otimes m=2}^{N-2} T(m)\, e^{(K_z^* - K_x)(-2i\eta_1(0)\eta_2(0))} \, e^{(K_z^* + K_x)(-2i\eta_1(\pi)\eta_2(\pi))} }$$

and $T(m)$'s expressions are as before but m takes even values.

(*) the τ- continum limit



Conside the limit $K_x \to 0$, set $K_x = \tau \to 0$.

and $K_z \to \infty$, such that $K_z^* = \lambda \tau \to 0$, where

λ is a constant. Then the two exponents

$$T = \exp\left[K_x \sum_{j=1}^{N} \sigma_z(j)\sigma_z(j+1) \right] \cdot \exp\left[\theta \sum_{j=1}^{N} \sigma_x(j) \right] \leftarrow \theta = K_z^*$$

$$= \exp\left[\tau \sum_{j=1}^{N} \sigma_z(j)\sigma_z(j+1) \right] \exp\left[\lambda\tau \sum_{j=1}^{N} \sigma_x(j) \right]$$

$$\simeq \exp[-\tau H] \quad \text{with} \quad \boxed{H = -\sum_{j=1}^{N} \left(\lambda \sigma_x(j) + \sigma_z(j)\sigma_z(j+1) \right)}$$

or in the Majarana Rep

$$H = \sum_{j=1}^{N} \left[i\lambda \psi_1(j)\psi_2(j) - i\psi_1(j)\psi_2(j+1) \right]$$

with the BC $\psi(N+1) = \mp \psi(1)$, for $(-)^{N_\psi} = \pm 1$.

Now let's treat $H = -\sum_i (\lambda \sigma_x(i) + \sigma_z(i)\sigma_z(i+1))$

as a quantum model, and consider it's ground state properties.

① strong coupling limit $\lambda \gg 1$

If $g \to \infty$, the ground state is a paramagnetic state with each site spin parallel to $\hat{x}$-direction.

$$|\Omega\rangle = \prod_i |\rightarrow\rangle_i \,, \quad \text{and} \quad \langle\Omega| \sigma_i^z \sigma_j^z |\Omega\rangle = \delta_{ij}.$$

If g is large but finite, we expect $\langle\Omega| \sigma_i^z \sigma_j^z |\Omega\rangle \sim e^{-|x_i - x_j|/\xi}$,

i.e. short-range correlated. The excitation is to flip one site spin to $\leftarrow$, i.e.

$$\rightarrow \rightarrow \cdots \underset{i}{\leftarrow} \rightarrow \rightarrow \qquad \rightarrow \qquad \boxed{|i\rangle = |\leftarrow\rangle_i \prod_{j\neq i} |\rightarrow\rangle_j}$$

All the states $|i\rangle$ are degenerate at the limit $\lambda \to +\infty$. At $1/\lambda$ level, the $\sigma_z \cdot \sigma_z$ term couples different states together as

$$\boxed{\langle i| -\sum_n \sigma_z(n)\sigma_z(n+1) |i\pm 1\rangle = -1}$$

we can form $|k\rangle = \frac{1}{\sqrt{N}} \sum_j e^{ikj} |j\rangle$. its eigen energy is

$$\boxed{\mathcal{E}_k = [2 - \tfrac{2}{\lambda}\cos(k) + O(1/\lambda^2) + \cdots]}$$

③ weak coupling $\lambda \ll 1$

two fold degeneracy $|\uparrow\rangle \otimes |\uparrow\rangle, \cdots$ and $|\downarrow\rangle \otimes |\downarrow\rangle \cdots \otimes |\downarrow\rangle$.

σ^z has long-range order. The low energy excited states are topological nature — kink.

$$|\uparrow\rangle \otimes |\uparrow\rangle \cdots \underset{i}{|\uparrow\rangle} \otimes \underset{i+1}{|\downarrow\rangle} \otimes |\downarrow\rangle \cdots$$

if we neglect the coupling between sectors with different number of kinks. we can easily work out its energy dispersion

$$\boxed{\varepsilon_k = (2 - 2\lambda \cos ka + O(\lambda^2))}$$

[the σ_1 term builds up hopping of kinks].

* Eigenvalues of the ~~transfer~~ matrix in the τ- continuum limit (13)

- In the Fermion # even sector

according to $\psi_\alpha(j) = \frac{1}{\sqrt{N/2}} \sum_{m=1,3,\cdots N-1} c_\alpha(k_m) e^{ik_m j} + c_\alpha^\dagger(k_m) e^{-ik_m j}$

$$\Rightarrow H_E = \sum_m 2i\lambda \left[c_1(k_m) c_2^\dagger(k_m) + c_1^\dagger(k_m) c_2(k_m) \right] - 2i \left[c_1(k_m) c_2^\dagger(k_m) e^{-ik_m} + c_1^\dagger(k_m) c_2(k_m) e^{ik_m} \right]$$

$k_m = \frac{m\pi}{N}$, $m = 1, 3, \cdots N-1$,

here $\eta^2(0) = \eta^2(\pi) = 1/2$.

- In the Fermion # odd sector

$$\psi_\alpha(j) = \frac{1}{\sqrt{N/2}} \left\{ \sum_{m=2,4\cdots}^{N-2} c_\alpha(k_m) e^{ik_m j} + c_\alpha^\dagger(k_m) e^{-ik_m j} + \eta_\alpha(0) + \eta_\alpha(\pi)(-)^j \right\}$$

$$H_0 = \sum_m \Big\{ 2i\lambda \left[c_1(k_m) c_2^\dagger(k_m) + c_1^\dagger(k_m) c_2(k_m) \right) - 2i \left[c_1(k_m) c_2^\dagger(k_m) e^{-ik_m} + c_1^\dagger(k_m) c_2(k_m) e^{ik_m} \right] \Big\}$$
$$+ 2i(\lambda-1)\eta_1(0)\eta_2(0) + 2i(\lambda+1)\eta_1(\pi)\eta_2(\pi)$$

$k_m = \frac{m\pi}{N}$, $m = 2, 4, \cdots N-2$

- The number of fermion number

$$N_\psi = \sum_{j=1}^{N} \Psi^\dagger(j)\Psi(j) = \sum_{j=1}^{N} \frac{1}{2}(1 + i\psi_1(j)\psi_2(j))$$

$$= \sum_1^N \frac{1}{2} + \sum_{m=1,3,\cdots N-1} i\left[c_1(k_m) c_2^\dagger(k_m) + c_1^\dagger(k_m) c_2(k_m) \right] \leftarrow \text{for even sector}$$

$$= \sum_1^N \frac{1}{2} + \sum_{m=2,4,\cdots N-2} i\left[c_1(k_m) c_2^\dagger(k_m) + c_1^\dagger(k_m) c_2(k_m) \right] + i\left[\eta_1(0)\eta_2(0) + \eta_1(\pi)\eta_2(\pi) \right]$$

$\leftarrow$ for odd sector

Nevertheless. we only interested in $(-)^{N_\psi}$. We can drop $\sum_1^N \frac{1}{2} = \frac{N}{2}$, since N is a multiplet of 4. Let's work in the basis of $C_1^\dagger C_1$ and $C_2^\dagger C_2$.

$$i\,[C_1(k_m)\, C_2^\dagger(k_m) + C_1^\dagger(k_m)\, C_2(k_m)]\ |00\rangle = i\,[C_1(k_m) C_2^\dagger(k_m) + C_1^\dagger(k_m) C_2(k_m)]\,|11\rangle = 0$$

$$\left.\begin{aligned} i\,[C_1(k_m)\, C_2^\dagger(k_m) + C_1^\dagger(k_m)\, C_2(k_m)]\,|01\rangle &= i\,|10\rangle \\ i\,[C_1(k_m)\, C_2^\dagger(k_m) + C_1^\dagger(k_m)\, C_2(k_m)]\,|10\rangle &= -i\,|01\rangle \end{aligned}\right\}\ \text{like } \tau_y$$

$\Rightarrow$ $e^{i\pi\,[i(C_1(k_m) C_2^\dagger(k_m) + C_1^\dagger(k_m) C_2(k_m))]}$ behaves 1 in the sector of $|00\rangle$ and $|11\rangle$, and behaves $e^{i\pi\tau_y} = -1$ in the sector of $|01\rangle$ and $|10\rangle$.

Hence $(-)^{N_\psi} = \begin{cases} (-)^{N_c} & \text{for even sector} \\ (-)^{N_c'} \cdot (-)^{i[\eta_1(0)\eta_2(0) + \eta_1(\pi)\eta_2(\pi)]} & \text{for the odd sector} \end{cases}$

here N_c' means excluding the modes with $k = 0, \pi$.

(*) Ground state energy

(1) in the even (fermion #) sector

$$H_E = \sum_{m>0} \left(C_1^\dagger(k_m)\ C_2^\dagger(k_m)\right) \begin{pmatrix} 0, & 2i(\lambda - e^{ik_m}) \\ -2i(\lambda - e^{-ik_m}) & 0 \end{pmatrix} \begin{pmatrix} C_1(k_m) \\ C_2(k_m) \end{pmatrix}$$

define $\begin{pmatrix} c_1(k_m) \\ c_2(k_m) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -e^{i\theta} \\ e^{-i\theta} & 1 \end{pmatrix} \begin{pmatrix} \eta_+(k_m) \\ \eta_-(k_m) \end{pmatrix}$ we have

$$H_E = \sum_{k_m>0} \varepsilon(k_m) \left[\eta_+^\dagger(k_m)\eta_+(k_m) - \eta_-^\dagger(k_m)\eta_-(k_m) \right]$$

with $\varepsilon(k_m) = 2\left(\lambda^2 + 1 - 2\lambda\cos k\right)^{1/2}$, $k_m = \frac{m\pi}{N}$, $m = 1, 3, \cdots N-1$.

we define $\begin{cases} \eta(k_m) = \eta_+(k_m) \\ \eta(-k_m) = \eta_-^\dagger(k_m) \end{cases}$ for $k_m > 0$, then

$$H_E = \sum_{k_m>0} \varepsilon(k_m)\left(\eta^\dagger(k_m)\eta(k_m)\right) + \sum_{k_m>0}\left(-\eta(-k_m)\eta^\dagger(-k_m)\right)\varepsilon(k_m)$$

$$= \sum_{k_m>0} \varepsilon(k_m)\left(\eta^\dagger(k)\eta(k_m)\right) + \sum_{k_m<0}\left(\eta^\dagger(k_m)\eta(k_m) - 1\right]\varepsilon(k_m)$$

$$= \sum_{k_m} \varepsilon(k_m)\left[\eta^\dagger(k_m)\eta(k_m) - \tfrac{1}{2}\right],$$ where now we have extend

$k_m = \pm\frac{m\pi}{N}$, $m = 1, 3, \cdots N-1$.

How about particle number?

It's to show $N_c = N_{\eta_+} + N_{\eta_-}$ since $c_{1,2}$ and $\eta_\pm$ are related by a unitary transformation. Then

$$N_{\eta_+} + N_{\eta_-} = \sum_{m>0} \eta_+^\dagger(k_m)\eta_+(k_m) + \eta_-^\dagger(k_m)\eta_-(k_m) = \sum_{m>0} \eta^\dagger(k_m)\eta(k_m) + \eta(-k_m)\eta^\dagger(-k_m)$$

$$= \sum_{m>0} \eta^\dagger(k_m)\eta(k_m) - \eta^\dagger(-k_m)\eta(-k_m) + 1 \equiv \sum_{m>0} \eta^\dagger(k_m)\eta(k_m) + \eta^\dagger(-k_m)\eta(-k_m) \pmod 2$$

The ground state in the even sector, can be achieved by simply leaving all η_m' modes empty, then

$$\boxed{E_E^{min} = -\frac{1}{2}\sum_{\substack{k_m \\ =\pm \frac{m\pi}{N}}} \varepsilon(k_m) = -\sum_{k_m} \sqrt{1-2\lambda\cos k_m + \lambda^2}\,, \quad \text{with } m=1,3,\cdots N-1.}$$

As for the Odd sector

$$H_O = \sum_{k_m \neq 0,\pi} [\eta^\dagger(k_m)\eta(k_m) - 1/2]\,\varepsilon(k_m) + 2i(\lambda-1)\eta_1(0)\eta_2(0) + 2i(\lambda+1)\eta_1(\pi)\eta_2(\pi)$$

defining $\eta(0) = \frac{1}{\sqrt{2}}[\eta_1(0) + i\eta_2(0)]$, $\eta(\pi) = \frac{1}{\sqrt{2}}[\eta_1(\pi) + i\eta_2(\pi)]$

$$\Rightarrow \quad 2i(\lambda-1)\eta_1(0)\eta_2(0) + 2i(\lambda+1)\eta_1(\pi)\eta_2(\pi) = (\eta^\dagger(0)\eta(0) - 1/2)\,\varepsilon(k=0) + (\eta^\dagger(\pi)\eta(\pi) - 1/2)\,\varepsilon(k=\pi)$$

with $\boxed{\varepsilon(0) = 2(\lambda-1)}$ and $\varepsilon(\pi) = 2(\lambda+1)$.

Now check Fermion parity

$$(-)^{N_f} = (-)^{N_c'}\,(-)^{i[\eta_1(0)\eta_2(0) + \eta_1(\pi)\eta_2(\pi)]} = (-)^{N_c'}\,(-)^{(\eta_0^\dagger\eta_0 + \eta_\pi^\dagger\eta_\pi - 1)}$$
$$= (-)^{\sum_{k_m\neq 0,\pi} \eta_{k_m}^\dagger \eta_{k_m}}\,(-)^{\eta_0^\dagger\eta_0 + \eta_\pi^\dagger\eta_\pi - 1}$$

$$= (-)^{(\sum_{k_m} \eta^\dagger(k_m)\eta(k_m) - 1)}$$

Hence $(-)^{N_f}$ odd, also means that $\eta^\dagger\eta$ number needs to be even.

However, please pay attention that π-mode can only be counted once.

since all the modes except $k=0$, contributes a negative energy, they should be occupied, and this will contribute $k=\pi, \pm \frac{m}{N}\pi$ $(m=2,4,\cdots N-2)$.

$\Rightarrow$ there're only even η-particles. Hence the $\eta(k=0)$ mode need to be occupied!

But the energy of this mode can be either positive ($\lambda>1$) or negative at $\lambda<1$, respectively. This gives rise to the phase transition!

(*) Asymptotic degeneracy at $\lambda<1$.

Odd sector

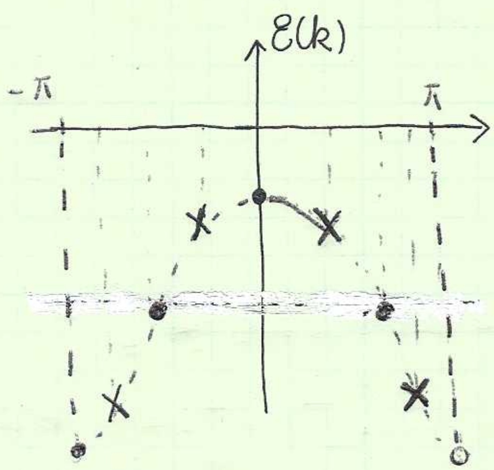
At $\lambda<1$, $\varepsilon(k=0)=\lambda-1<0$. Hence filling it contributes negatively.

$$E_0^{min} = -\frac{1}{2}\sum_{k_m} \varepsilon(m) = -\sum_{k_m}\sqrt{1-2\lambda\cos k_m+\lambda^2}$$ with $k_m=0$. included

$$k_m = 0, \pm \frac{m\pi}{N}, \pi, \quad \text{with } m=2,4,\cdots N-2.$$

Compare with the even sector, $\rightarrow$ the same expression of $\varepsilon(k_m)$, but

$$k_m = \pm\frac{m\pi}{N} \quad \text{with } m=1,3,\cdots N-1.$$



which one is the true ground state?

conside $\Gamma_5 = \sigma_x\otimes\sigma_x\cdots\otimes\sigma_x$, which flips all the spin to its opposite. Since the transverse field term $H=-\lambda\sum_j \sigma_x(j)$, the even secter of Γ_5 have better energy.

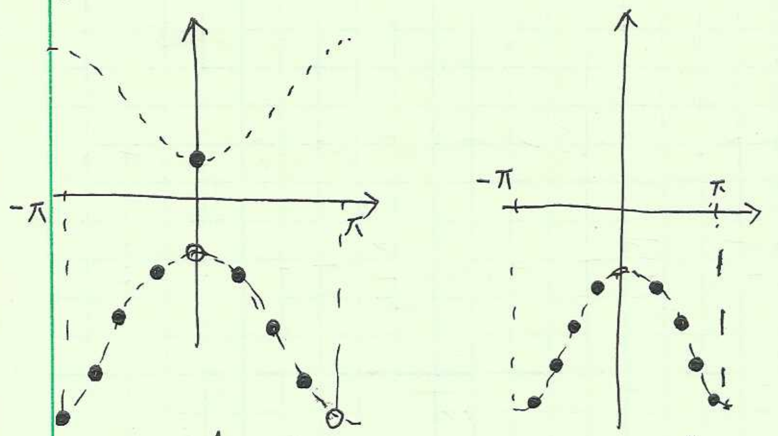
$$|\text{even}\rangle = \frac{|\text{all up}\rangle + |\text{all down}\rangle}{\sqrt{2}} \quad\Rightarrow\quad \Gamma_5\,|\text{even(odd)}\rangle = \pm\,|\text{even(odd)}\rangle$$

$$|\text{odd}\rangle = \frac{|\text{all up}\rangle - |\text{all down}\rangle}{\sqrt{2}}$$

The energy difference is exponetially samall between those in the even and odd sectors.

$\boxed{\text{Ex:}}$ please check based on the dispersion above, and verify the energy difference $\Delta E = E_{min,o} - E_{min,e} \to O(e^{-N})$.

⊛ ground state at $\lambda > 1$.



odd sector　　　　even sector

there're only a unique ground state in the even sector.
→ "high T region", disordered.

⊛ Free energy in the thermodynamic limit

$$T = e^{-H\tau}, \quad \text{and} \quad Z = \lim_{M\to\infty} \mathrm{Tr}\, T^M \longrightarrow \lambda_0^M = e^{-M E_0 \tau}$$

where λ_0 is the largest eigenvalue of T, or E_0 is the smallest eigenvalue of H. Then the free energy per site

$$-\beta f \to \frac{1}{MN} \ln Z = -\frac{1}{N} E_0 \tau .$$

M: site along time domain
N: site along space domain

at $\lambda<1$, there're two degenerate ground state

$Z = 2e^{-ME_0\tau}$, the degeneray 2 can be neglected when calculate $\frac{\ln Z}{MN}$

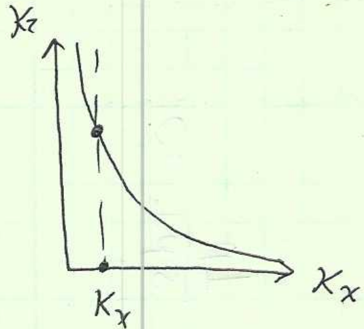
· Now $\beta f = \frac{1}{N}E_0\tau = -\frac{\tau}{N}\sum_{k_m}\frac{1}{2}\varepsilon(k_m)$ ← plug in $\tau = K_x$

$$= -K_x\int_{-\pi}^{\pi}\frac{dk}{2\pi}\sqrt{1-2\lambda\cos k+\lambda^2}$$

(we neglect the prefacters $(\cosh K_\tau^*)^N$, which is smooth, no contribut to phase transition)

Since ~~temperature~~ enters the coupling K_x and K_τ

and $\lambda = \frac{K_\tau^*}{K_x}$



At $T_c \Rightarrow$ K_{x_c} and K_{τ_c} satisfy

$$\sinh 2K_{x_c}\sinh 2K_{\tau_c} = 1 \Rightarrow K_{\tau,c} = K^*_{x,c}$$

hence $\lambda_c = \frac{K^*_{\tau_c}}{K_{x_c}} = \frac{(K^*_{x_c})^*}{K_{x_c}} = \frac{K_{x_c}}{K_{x_c}} = 1$

As $T \to T_c + \Delta T$, and define $t = \Delta T/T_c$, we have

$$K_{x_c} \longrightarrow K_{x_c}T_c/(T_c+\Delta T) = K_{x_c}(1-t)$$

$$K_{\tau_c} \longrightarrow K_{\tau_c}\,T_c/(T_c+\Delta T) = K_{\tau_c}(1-t)$$

$-\cosh 2K_\tau \, \Delta K_\tau$

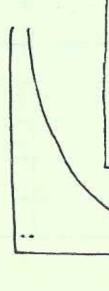
Then $$\lambda = \frac{K_\tau^*}{K_x} = \frac{[K_{\tau_c}(1-t)]^*}{K_{x_c}(1-t)} = \frac{(K^*_{x_c}(1-t))^*}{K_{x_c}(1-t)} \approx \frac{K_{x_c}^{(1-t)}}{K_{x_c}(1-t)} \approx 1+(1-\ln K_{x_c})t$$

$\boxed{\text{hence } \lambda - 1 \propto t \text{ scales linearly!}}$

~~Since $K_\tau = K_x^* \to \infty$, hence~~

it's easy to show, as $x \to 0$, $x^* \to \frac{1}{2}\ln 1/x$

as $x \to \infty$, $x^* \to e^{-2x}$



Now let us extract the singular part of the free energy

$$f_s \propto -\int_0^{\pi} \sqrt{(\lambda-1)^2 + 2\lambda(1-\cos k)}\, dk \sim -\int_0^{\pi} \sqrt{t^2+k^2}\, dk$$

↓ drop coefficient

set $k = t\,\mathrm{sh}\,x$, then $\int \sqrt{t^2+k^2}\, dk = t^2 \int \mathrm{ch}^2 x\, dx = \frac{t^2}{2}\int (1+\mathrm{ch}\,2x)\, dx$

$$= \frac{t^2}{2}\left[x + \frac{1}{2}\mathrm{sh}\,2x\right] = \frac{t^2}{2}\mathrm{sh}^{-1}\frac{k}{t} + \frac{k}{4}\sqrt{t^2+k^2}$$

$$-\int_0^{\Delta} \sqrt{t^2+k^2}\, dk \simeq -\frac{t^2}{2}\mathrm{sh}^{-1}\frac{\Delta}{t} + \text{regular terms}, \quad \leftarrow \mathrm{sh}^{-1} x = \ln(x + \sqrt{x^2-1})$$

$$\xrightarrow[t\to 0]{} \simeq -\frac{t^2}{2}\ln\frac{2\Delta}{t} \simeq -t^2 \ln \Delta/|t|$$

Hence $C_V = +T\frac{dS}{dT} = -T\frac{d}{dT}\left(\frac{dF}{dT}\right) \propto -\frac{d^2 f}{dt^2} \propto \ln\frac{\Delta}{|t|}$

Now we have established the famous result of logrithimic peak of C_V.

Hence the critical exponent $\alpha = 0^+$.

(*) Correlation functions

It's easier to consider the correlation function along the time domain

$$G(n\tau) = \langle \sigma_x(0, n\tau)\, \sigma_x(0,0)\rangle \leftarrow \text{the onsite but displaced along time } n\tau$$

$$= \sum_j |\langle j|\sigma_x|vac\rangle|^2 e^{-E_j n \tau}, \quad \text{where } j \text{ is excited states}$$

$\sigma_x = -2i\psi_1^* \psi_2$, it will correspond to excite two η-particles

the lowest energy $4|\lambda-1|$, $\Rightarrow G(n\tau) \rightarrow e^{-4|\lambda-1|n\tau} = e^{-n\tau/\xi(t)}$

with $\xi(t) \sim \frac{1}{|t|}$, i.e $\nu = 1$.

Please note that this power is far from the mean field value $\nu = 1/2$.

At mean field GL free energy, the propogator $\sim \frac{1}{k^2 + |t|}$, hence $\xi \sim |t|^{-1/2}$.

Now the system becames a fermion theory $\omega \sim \sqrt{k^2 + |t|^2} \Rightarrow \xi \sim |t|^{-1}$.

(*) Magnetization

$$|\langle M \rangle| = \left[1 - \frac{1}{\sinh^2 2K_x \sinh^2 2K_\tau} \right]^{1/8} \quad \text{— Onsager, C.N. Yang}$$

- At the phase transition. $\sinh^2 2K_x \sinh^2 2K_\tau = 1$, $\langle M \rangle = 0$.
- plugin $K_x = \tau$, $K_\tau^* = \lambda\tau$, at $\tau \to 0$,

then $K_\tau = (K_\tau^*)^* \sim \frac{1}{2} \ln \frac{1}{\lambda\tau}$, hence $\sinh^2 2K_x \sim (2\tau)^2$

$$\sinh^2 2K_\tau \sim \left(\frac{1}{2} e^{\ln \frac{1}{\lambda\tau}}\right)^2 \sim \frac{1}{4}\left(\frac{1}{\lambda\tau}\right)^2$$

$$|\langle M \rangle| \sim \left|1 - \frac{1}{\lambda}\right|^{1/8} \xrightarrow[\lambda \to 1]{\simeq} |(1-\lambda)|^{1/8}, \text{ i.e. } \beta = 1/8.$$

(*) anomalous dimension $\eta = 1/4$ — difficult, let us wait.

(*) $$\chi(t) = \int d^2 r\, G(r) \sim \int r\, dr \frac{e^{-r/\xi(t)}}{r^{d-2+\eta}} = \int r\, dr \frac{e^{-r/|t|}}{r^{1/4}}$$

$$\sim |t|^{-7/4} \int_0 x^{3/4} e^{-x} dx \quad \text{hence} \quad \boxed{\gamma = 7/4}$$

othe exponents

$\alpha = 0^+$, $\beta = 1/8$, $\gamma = 7/4$, $\delta = 15$, $\nu = 1$, $\eta = 1/4$.

Another version of Jordan-Wigner transformation

Define non-local transformation: Jordan-Wigner transformation

$$\sigma_i^z = 1 - 2c_i^\dagger c_i$$

$$\sigma_i^+ = \prod_{j<i}(1-2c_j^\dagger c_j)\,c_i \qquad \xrightarrow{\text{inverse}} \qquad c_i = \left(\prod_{j<i}\sigma_j^z\right)\sigma_i^+$$

$$\sigma_i^- = \prod_{j<i}(1-2c_j^\dagger c_j)\,c_i^\dagger \qquad\qquad c_i^\dagger = \left(\prod_{j<i}\sigma_j^z\right)\sigma_i^-$$

Ex: please check that $\{c_i, c_j^\dagger\} = \delta_{ij}$, and thus $c_i, c_i^\dagger$ are spinless fermion operators.

For transverse field Ising model, it's more convenient to do a further transform $\sigma^z \to \sigma^x$ and $\sigma^x \to -\sigma^z$.

Such that

$$\sigma_i^x = 1 - 2c_i^\dagger c_i$$

$$\sigma_i^z = -\prod_{j<i}(1-2c_j^\dagger c_j)(c_i + c_i^\dagger)$$

$$\Rightarrow \quad H = -K\sum_i \{ g(1-2c_i^\dagger c_i) + (c_i + c_i^\dagger)(c_{i+1} + c_{i+1}^\dagger) \}$$

$$= -K\sum_i \left(c_i^\dagger c_{i+1} + c_{i+1}^\dagger c_i + c_i^\dagger c_{i+1}^\dagger + c_{i+1} c_i - 2g c_i^\dagger c_i - g \right)$$

$$= K\sum_k \left(2(g-\cos k) c_k^\dagger c_k - 2i\sin k \left(c_{-k}^\dagger c_k^\dagger - c_k c_{-k} \right) - g \right]$$

$$= K\sum_k \begin{pmatrix} c_k^\dagger & c_{-k} \end{pmatrix} \begin{bmatrix} 2(g-\cos k) & 2i\sin k \\ -2i\sin k & -2(g-\cos k) \end{bmatrix} \begin{bmatrix} c_k \\ c_{-k}^\dagger \end{bmatrix}$$

$\rightarrow$ The excitation spectrum

$$\varepsilon_k = 2K\left((g-\cos k)^2 + \sin^2 k \right)^{1/2} = 2K\left(1 + g^2 - 2g\cos k\right)^{1/2}$$

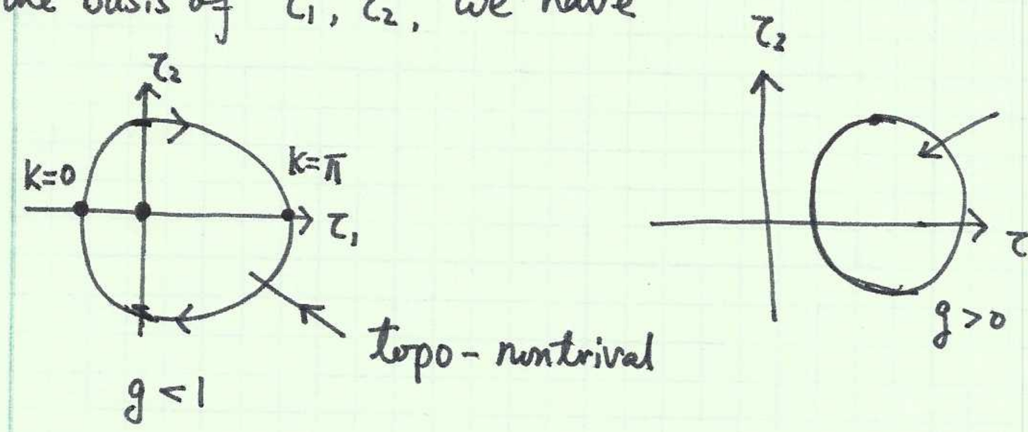
Ex: ① please diagonalize the above matrix by Bologolibov transformation

② check that ε_k at $g \ll 1$ and $g \gg 1$, agrees with the approximate expression given above.

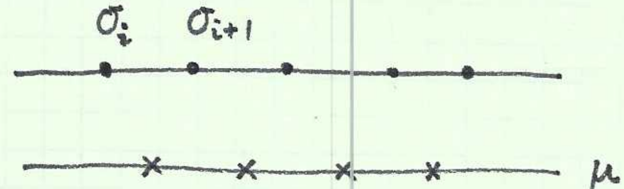
At both $g>1$, and $g<1$, because $1+g^2 > 2g$, the spectra of ε_k is gapped. But at $g=1$, $\varepsilon_k = 4K\left|\sin\frac{k}{2}\right|$, the spectra is gapless, which indicate a quantum phase transition. Indeed, $|g|<1$ corresponds to topological pairing, and $|g|>1$ is topologically trivial pairing!

The pairing matrix $\Delta_k = 2[(g-\cos k)\tau_1 - \sin k\, \tau_2]$

as k in the BZ. $k \in [-\pi, \pi]$, if we represent Δ_k as a 2-vector in the basis of τ_1, τ_2, we have



& come back to the spin language, we have an order / disorder transition.



Duality (site-bond)

$$\begin{cases} \mu^z_{n+1/2} = \prod_{j=1}^{n} \sigma^x_j \\ \mu^x_{n+1/2} = \sigma^z_n \sigma^z_{n+1} \end{cases} \longrightarrow \begin{cases} \sigma^z_n = \prod_{j=0}^{n-1} \mu^x_{j+1/2} \\ \sigma^x_n = \mu^z_{n-1/2} \mu^z_{n+1/2} \end{cases}$$

in terms of $\mu \Rightarrow$

$$H = -K\left[g \sum_n \mu^z_{n-1/2} \mu^z_{n+1/2} + \mu^x_{n+1/2} \right]$$

$g \to 1/g$. self-duality.

What is μ? the Kink operator / disorder operator

$$|\Omega\rangle = \prod_n |\uparrow\rangle_n \Rightarrow \mu^z_{n+1/2}|vac\rangle = |\underset{n}{\downarrow\downarrow\cdots\downarrow}\underset{n+1/2}{\uparrow}\uparrow\uparrow\cdots\rangle$$

Thus $g>1$, σ_z disordered, $\longleftrightarrow$ μ^z ordered

<1 σ_z ordered $\longleftrightarrow$ μ^z disordered

self-dual

Further Come back to 2D Ising model $\Longrightarrow$ low $T<T_c$ / $T>T_c$ Winger-Kramer duality.

§ Majarana Representation

$$\xi_1(n) = \frac{C_n^\dagger + C_n}{\sqrt{2}}, \quad \xi_2(n) = \frac{C_n^\dagger - C_n}{-\sqrt{2}\, i} \Rightarrow \{\xi_i, \xi_j\} = \delta_{ij}$$

Ex: please verify that in the Majorana Rep

$$H = K(\ ig\, \xi_2(n)\xi_1(n) - i\xi_2(n)\xi_2(n+1)]$$

$\rightarrow$ antinumse version

$$\frac{H}{K} = -i\xi_2(n)(\xi_1(n+1) - \xi_1(n)) + i(g-1)\xi_2(n)\xi_1(n)$$

$$\rightarrow \int dx\ \xi_2(-i\partial_x)\xi_1 - im\xi_1\xi_2 \qquad m = g-1$$

$$= \frac{1}{2}\int dx\ \xi^T(\alpha p + \beta m)\xi, \quad \text{where } \xi = \begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix}, \quad \alpha = \begin{pmatrix} & 1\\ 1 & \end{pmatrix}$$

$$\beta = \begin{pmatrix} & -i\\ i & \end{pmatrix} \quad p = -i\partial_x$$