

θ-term Haldane conjecture

①

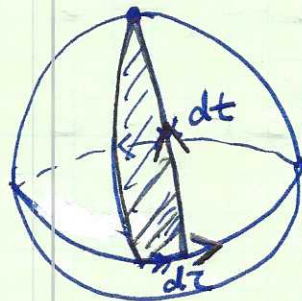
- Wess-Zumino term for a single spin

$$Z = \int D\hat{n}(z) e^{-S(\hat{n})}, \quad \text{where } S(\hat{n}) = iS\omega[\hat{n}(\omega)] + \int_0^\beta dz H(\hat{n})$$

$$\omega[\hat{n}(\omega)] = \int d\phi (1 - \cos\theta) = \int d\hat{n} \cdot \vec{A}(\hat{n})$$

$$= \int_0^\beta dz \int_0^1 dt \hat{n}(\tau, t) \cdot (\partial_z \hat{n}(\tau, t) \times \partial_t \hat{n}(\tau, t))$$

$$\text{with } \begin{cases} \vec{n}(\tau, 0) = \hat{n}(\tau) \\ \vec{n}(\tau, 1) = \hat{n}_0 \end{cases}$$



Note the WZ term does not change when back to the real time

$$S_M = S\omega[\hat{n}(\tau)] - \int_0^T dt H(\hat{n}) \rightarrow Z = \int D\hat{n} e^{iS_M[\hat{n}]}$$

- Now we derive the action for a 1D spin chain (AF)

Due to the AF tendency, we separate slow mode and the fast mode

$$\hat{n}(j) = \hat{m}_j(\tau) + \vec{l}(j) a_0$$

← lattice constant

$$|\hat{m}|^2 = 1, \text{ and } \hat{m} \cdot \vec{l} = 0, \text{ with } |\vec{l}| \ll 1.$$

$$S_M(\hat{n}) = S \sum_{j=1}^N \omega[\hat{n}(j)] - \int_0^T dx_0 \sum_{j=1}^N JS^2 \hat{n}(j, x_0) \cdot \hat{n}(j+1, x_0)$$

$$\rightarrow -\frac{JS^2}{2} \int_0^T dx_0 \sum_{j=1}^N (\hat{n}(j, x_0) + \hat{n}(j+1, x_0))^2$$

$$= -\frac{JS^2}{2} \int_0^T dx_0 \sum_{j=1}^N [(\hat{m}(j, x_0) - \hat{m}(j+1, x_0))(-)^l + a_0(\vec{l}(j, x_0) + \vec{l}(j+1, x_0))]^2$$

$$\downarrow \qquad \qquad \qquad \downarrow$$

$$(\partial_t \hat{m}) \cdot a_0 (-)^l \qquad \qquad \qquad 2a_0 \vec{l}$$

$$= -\frac{a_0 JS^2}{2} \int_0^T dx_0 \int dx [(\partial_t \hat{m})^2 + 4\vec{l}^2] \leftarrow \sum_j = \frac{1}{a_0} \int dx$$

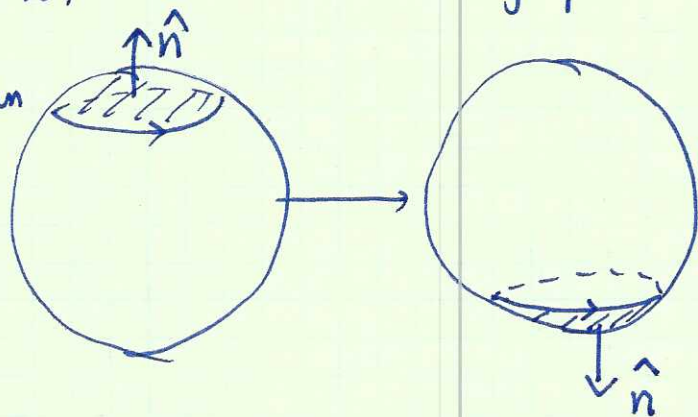
HW: Show that the cross term vanishes after summing over "i".

Hint: $(\partial_t \hat{m})(-)^l$ and \vec{l} belong to different sectors of

momenta. One is staggered, the other is uniform.

Now check the W-Z term:

If we flip the direction of \hat{n} , the WZ term also flips the sign. The circulation orientation does not change, but the norm change, hence the areas surrounded have opposite sign.

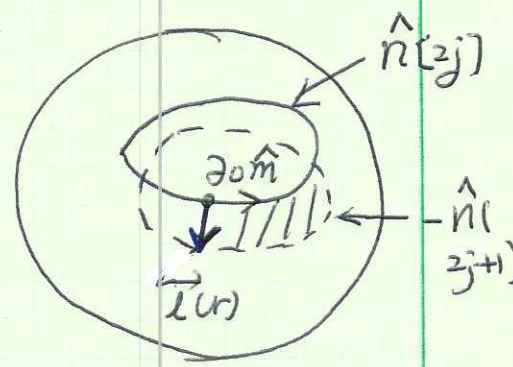


Hence the WZ term from nearest neighbors nearly cancel.

$$S \sum_{j=1}^N W(n[j, z]) = S \sum_{j=1}^{N/2} \omega[n(z_j)] - \omega[-n(z_{j+1})]$$

$$\omega[n(z_j)] - \omega[-n(z_{j+1})]$$

$$= \int dx_0 [n(z_j) + n(z_{j+1})] \cdot \underline{\partial_0 \hat{n}(z_j) \times \hat{n}(z_j)}$$



$$= S \int dx_0 [\partial_1 m(j) \cdot a_0 + 2\vec{l}(j) \cdot a_0] \cdot (\partial_0 \hat{m}(z_j) \times \hat{m}(z_j))$$

For term underlined, we could also use $(-\partial_0 \hat{n}(z_{j+1}) \times (-\hat{n}(z_{j+1})))$
 $\sim (\partial_0 \hat{m} - \hat{l}) \times (\hat{m} - \hat{l})$

while $\partial_0 \hat{n}(z_j) \times \hat{n}(z_j) \sim (\partial_0 \hat{m} + \hat{l}) \times (\hat{m} + \hat{l})$

as an average for high order accuracy, $\rightarrow (\partial_0 \hat{m} \times \hat{m})$

$$\Rightarrow S \sum_{j=1}^N \omega[n(j, z)] = S \sum_{j=1}^{N/2} \int dx_0 [\partial_1 \hat{m} \cdot a_0 + 2\vec{l} \cdot a_0] \cdot (\partial_0 \hat{m} \times \hat{m})$$

$$= \frac{S}{2} \int dx_0 dx_1 \hat{m} \cdot (\partial_0 \hat{m} \times \hat{m}) - S \int dx_0 dx_1 \vec{l} \cdot (\hat{m} \times \partial_0 \hat{m})$$

$$\sum_{j=1}^{N/2} \cdot \rightarrow \int \frac{dx}{2a_0}$$

$$\Rightarrow \mathcal{L}_M(\vec{m}, \vec{l}) = -2a_0 J S^2 l^2 - \frac{a_0 J S^2}{2} (\partial_1 \hat{m})^2$$

$$- S \vec{l} \cdot (\hat{m} \times \partial_0 \hat{m}) + \frac{S}{2} \hat{m} \cdot (\partial_0 \hat{m} \times \hat{m})$$

$$Z_M = \int D\hat{m} D\ell \quad e^{i \int dt L_M(\hat{m}, \vec{\ell})}$$

$$\text{L term: } -2a_0 J S^2 \left[\vec{\ell} + \frac{\hat{m} \times \partial_0 \hat{m}}{4a_0 J S} \right]^2 + \frac{1}{8a_0 J} [\hat{m} \times \partial_0 \hat{m}]^2$$

$$\Downarrow \frac{1}{8a_0 J} (\partial_0 \hat{m})^2$$

$$\Rightarrow L_M(\hat{m}) = \frac{1}{8a_0 J} (\partial_0 \hat{m})^2 - \frac{a_0 J S^2}{2} (\partial_1 \hat{m})^2 + \frac{S}{2} \hat{m} \cdot (\partial_0 \hat{m} \times \partial_1 \hat{m})$$

$$\text{define } g = 2/8, \quad v_s = 2a_0 J S, \quad \theta = 2\pi S$$

$$\Rightarrow L_M = \frac{1}{2g} \left[\frac{1}{v_s} (\partial_0 \hat{m})^2 - v_s (\partial_1 \hat{m})^2 \right] + \theta \frac{\epsilon_{\mu\nu} \hat{m} \cdot (\partial_\mu \hat{m} \times \partial_\nu \hat{m})}{8\pi}$$

→ Euclidean action

$$L_E = \frac{1}{2g} \left[\frac{1}{v_s} (\partial_0 \hat{m})^2 + v_s (\partial_1 \hat{m})^2 \right] + \frac{i\theta}{8\pi} \epsilon_{ij} \hat{m} \cdot (\partial_i \hat{m} \times \partial_j \hat{m})$$

$$\int dx_0 \int dx_1 \frac{\epsilon_{\mu\nu}}{8\pi} \hat{m} \cdot (\partial_\mu \hat{m} \times \partial_\nu \hat{m}) = \text{Pontryagin } \#$$

$$\Rightarrow Z_M = \sum_n \int D[\hat{m}] \quad e^{i2\pi S \cdot n} \quad e^{-\int_0^\beta dz \frac{1}{2g} \left[\frac{1}{v_s} (\partial_0 \hat{m})^2 + v_s (\partial_x \hat{m})^2 \right]}$$

• for half-integer S

$$e^{i\pi 2Sn} = (-1)^n \Rightarrow Z_M = \sum_n \int D(\hat{m}) (-1)^n e^{-\int_0^\beta dz \frac{1}{2g} [\partial_\mu \hat{m} \cdot \partial_\mu \hat{m}]}$$

• ~~The~~ for integer spin, the Θ -term does not make a difference!

★ Haldan conjecture:

1. Spin 1/2 AFM Heisenberg model is gapless, and its non-linear σ model has non-trivial Θ -terms.

2. The classic ^{2D} non-linear σ model is disordered.

Conjecture: The gapless behavior of spin-1/2 AFM chain is due to effect of Θ -term. Hence, Haldane conjectured that half integer spin chains are all gapless. However, for integer AFM spin chain, they remain gapped!

But spin-1 AFM chain \neq classic 2D non-linear sigma model, it has boundary modes.

notes on non-linear σ model

$$\Omega = \rho_i \hat{n}(x) \sqrt{1 - \frac{L^2}{S^2}} + \frac{L^2}{S^2} \quad \vec{L} \cdot \vec{n} = 0$$

" \tilde{J} " only appears in the velocity, not in the coupling constant

$$L = \chi_0 (\partial_t n)^2 - \rho_s (\partial_x n)^2 \quad n^2 = 1$$

by definition $\rho_s = JS^2 a^{2-d}$ ~~and~~ check dimension

but χ comes from $J \cdot L^2 a^{-d} \Rightarrow \chi \propto \frac{1}{Ja^d}$

$$\Rightarrow v = \frac{\sqrt{\chi_0 \rho_s}}{S} \propto aJS$$

$$\theta = \frac{\sqrt{\chi_0 \rho_s}}{S} \propto Sa$$

coupling constant only contains S , no J

" \tilde{J} " effects on the temporal & spacial are opposite:

Large " \tilde{J} " means difficult to get ferromagnet, i.e. small susceptibility.

④ Spin-1 models (1D) - AKLT

$$H_{AKLT} = J \sum_i \vec{S}_i \cdot \vec{S}_{i+1} + \frac{1}{3} (\vec{S}_i \cdot \vec{S}_{i+1})^2 + \frac{2}{3}$$

$$\vec{S}_i \cdot \vec{S}_{i+1} = \frac{1}{2} (\vec{S}_i + \vec{S}_{i+1})^2 - 2 = \begin{cases} 1 & \text{for } S_{tot} = 2 \\ -1 & = 1 \\ -2 & = 0 \end{cases}$$

$$\rightarrow \vec{S}_i \cdot \vec{S}_{i+1} = P_2(i, i+1) - P_1(i, i+1) - 2P_0(i, i+1)$$

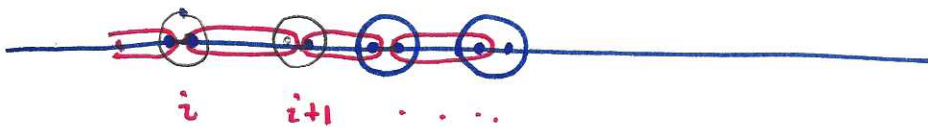
$$(\vec{S}_i \cdot \vec{S}_{i+1})^2 = P_2(i, i+1) + P_1(i, i+1) + 4P_0(i, i+1)$$

with $P_2(i, i+1) + P_1(i, i+1) + P_0(i, i+1) = 1$

$$\Rightarrow H_{AKLT} = J \sum_i P_2(i, i+1)$$

For spin-1, it can be written as 2-Schwinger bosons

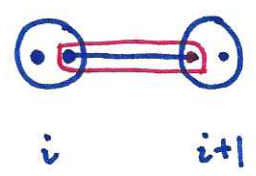
$S_z = 1$	$\frac{1}{\sqrt{2}} (a^\dagger)^2 0\rangle$	
0	$a^\dagger b^\dagger 0\rangle$	$a^\dagger a + b^\dagger b = 2$
-1	$\frac{1}{\sqrt{2}} (b^\dagger)^2 0\rangle$	



Consider state $|\psi\rangle = \prod_i (a_i^\dagger b_{i+1}^\dagger - b_i^\dagger a_{i+1}^\dagger) |0\rangle$

Then $|\psi\rangle$ is a ground state.

Consider a bond $i, i+1$. If we trace out the states of all other sites, we are left the configuration



$$(a_i^\dagger b_{i+1}^\dagger - b_i^\dagger a_{i+1}^\dagger) \otimes [\text{one boson on } i \text{ and one boson on } i+1]$$

$$a_i^\dagger b_{i+1}^\dagger - b_i^\dagger a_{i+1}^\dagger = (a_i^\dagger \ b_i^\dagger) \begin{pmatrix} 1 & \\ -1 & \end{pmatrix} \begin{pmatrix} a_{i+1}^\dagger \\ b_{i+1}^\dagger \end{pmatrix}$$

charge conjugation matrix R

$$R \begin{pmatrix} a^\dagger \\ b^\dagger \end{pmatrix} \text{ under rotation, transforms the same as } \begin{pmatrix} a \\ b \end{pmatrix}$$

Exercise prove that actually $(a_i^\dagger b_{i+1}^\dagger - b_i^\dagger a_{i+1}^\dagger)$ is invariant under

rotation, i.e. form singlet. \Rightarrow the total spin of sites $i, i+1$

only comes from one boson on i and one boson on $i+1$. The

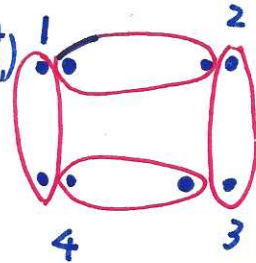
total spin can only be 0 or 1. \Rightarrow $P_2(i, i+1) |\psi\rangle = 0$.

\Rightarrow Since P_2 is a non-negative operator, $\Rightarrow |\psi\rangle$ is a ground state of H .

(I do not know how to prove the uniqueness of the ground state, I guess it is also true).

Let me use a 4-site ring to illustrate the WF

$$|\psi\rangle = (a_1^\dagger b_2^\dagger - b_1^\dagger a_2^\dagger)(a_2^\dagger b_3^\dagger - b_2^\dagger a_3^\dagger)(a_3^\dagger b_4^\dagger - b_3^\dagger a_4^\dagger)(a_4^\dagger b_1^\dagger - b_4^\dagger a_1^\dagger)$$



$$= a_1^\dagger b_2^\dagger a_2^\dagger b_3^\dagger a_3^\dagger b_4^\dagger a_4^\dagger b_1^\dagger - a_1^\dagger b_2^\dagger a_2^\dagger b_3^\dagger a_3^\dagger b_4^\dagger b_4^\dagger a_1^\dagger$$

L L L L L L L R

$$- a_1^\dagger b_2^\dagger a_2^\dagger b_3^\dagger b_3^\dagger a_4^\dagger a_4^\dagger b_1^\dagger + a_1^\dagger b_2^\dagger a_2^\dagger b_3^\dagger b_3^\dagger a_4^\dagger b_4^\dagger a_1^\dagger$$

L L R L L L R R

$$- a_1^\dagger b_2^\dagger b_2^\dagger a_3^\dagger a_3^\dagger a_4^\dagger b_1^\dagger + a_1^\dagger b_2^\dagger b_2^\dagger a_3^\dagger a_3^\dagger a_4^\dagger b_4^\dagger a_1^\dagger$$

L R L L L R L R

$$+ a_1^\dagger b_2^\dagger b_2^\dagger a_3^\dagger b_3^\dagger a_4^\dagger a_4^\dagger b_1^\dagger - a_1^\dagger b_2^\dagger b_2^\dagger a_3^\dagger b_3^\dagger a_4^\dagger b_4^\dagger a_1^\dagger$$

L R R L L R R R

$$- b_1^\dagger a_2^\dagger a_2^\dagger b_3^\dagger a_3^\dagger b_4^\dagger a_4^\dagger b_1^\dagger + b_1^\dagger a_2^\dagger a_2^\dagger b_3^\dagger a_3^\dagger b_4^\dagger b_4^\dagger a_1^\dagger$$

R L L L R L L R

$$+ b_1^\dagger a_2^\dagger a_2^\dagger b_3^\dagger b_3^\dagger a_4^\dagger a_4^\dagger b_1^\dagger - b_1^\dagger a_2^\dagger a_2^\dagger b_3^\dagger a_4^\dagger b_4^\dagger a_1^\dagger$$

R L R L R L R R

$$+ b_1^\dagger a_2^\dagger b_2^\dagger a_3^\dagger a_3^\dagger b_4^\dagger a_4^\dagger b_1^\dagger - b_1^\dagger a_2^\dagger b_2^\dagger a_3^\dagger a_3^\dagger b_4^\dagger a_4^\dagger b_1^\dagger$$

R R L L R R L R

$$- b_1^\dagger a_2^\dagger b_2^\dagger a_3^\dagger b_3^\dagger a_4^\dagger a_4^\dagger b_1^\dagger + b_1^\dagger a_2^\dagger b_2^\dagger a_3^\dagger b_3^\dagger a_4^\dagger b_4^\dagger a_1^\dagger$$

R R R L R R R R

$$\begin{aligned}
&= 2|0000\rangle - \frac{1}{2}|100-1\rangle - \frac{1}{2}|00-11\rangle + \frac{1}{2}|10-10\rangle \\
&\quad - \frac{1}{2}|0-110\rangle + \frac{1}{4}|1-11-1\rangle + \frac{1}{2}|0-101\rangle - \frac{1}{2}|1-100\rangle \\
&\quad - \frac{1}{2}|1-1100\rangle + \frac{1}{2}|010-1\rangle + \frac{1}{4}|-11-11\rangle - \frac{1}{2}|01-10\rangle \\
&\quad + \frac{1}{2}|-1010\rangle - \frac{1}{2}|001-1\rangle - \frac{1}{2}|-1001\rangle
\end{aligned}$$

Pattern observed ① If we remove "0", \Rightarrow then we have

1 -1 or 1-1-1, on a Neel order.

The AKLT ground state can be viewed by adding zeros into the Neel configure.

② states with more "1" and "-1" have less weight

HW ① For the long chain, prove that ^{for} in the ground state, in the S_z -representation, there does not exist two 1's or two -1's adjacent after 0's are removed.

② $O_{string} = \lim_{|i-j| \rightarrow \infty} \langle S_i^\alpha e^{i\pi \sum_{k=i}^{j-1} S_k^\alpha} S_j^\alpha \rangle \quad \alpha = x, y, z$

~~This non-local order develop nonzero expectation value.~~

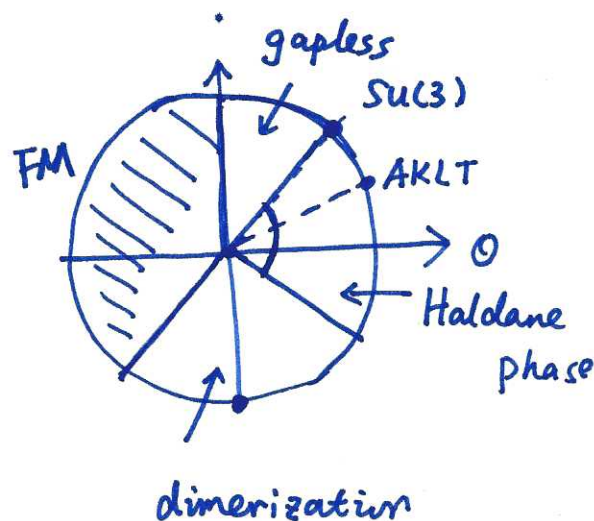
$O_{string} \neq 0 \rightarrow$ non-local order

~~AKLT state~~

Spin-1 mode $H = J \sum_i \cos \theta \vec{S}_i \cdot \vec{S}_{i+1} + \sin \theta (\vec{S}_i \cdot \vec{S}_{i+1})^2$

So $\theta = 0$, the pure Heisenberg model lies in the same phase of the AKLT one

	$S_i \cdot S_j$	$(S_i \cdot S_j)^2$	
Singlet 0	-2	4	$-2\cos \theta + 4\sin \theta$
Triplet 1	-1	1	$-\cos \theta + \sin \theta$
Quintet 2	1	1	$\cos \theta + \sin \theta$



① at $\theta = 45^\circ \Rightarrow E_{\text{singlet}} = E_{\text{quint}}$

$SU(3)$ symmetry $3 \times 3 = 3 + 6$

② at ~~$\theta = 45^\circ$~~ $\theta = -90^\circ$, $E_{\text{triplet}} = E_{\text{quint}}$ $\square \times \square^* = 1 + 8$

$SU(3)$, symmetry

{ other AKLT phase

