path integral for quantum spin

§ Schwinger boson representation

$$\boxed{S_x = \frac{a_1^\dagger a_2 + a_2^\dagger a_1}{2}, \quad S_y = \frac{a_1^\dagger a_2 - a_2^\dagger a_1}{2i}, \quad S_z = \frac{a_1^\dagger a_1 - a_2^\dagger a_2}{2}}$$

the value of S is implemented through the constraint $a_1^\dagger a_1 + a_2^\dagger a_2 = 2S$.

The normalized spin eigenstate $|S.m\rangle$ is represented as

$$|Sm\rangle = \frac{(a_1^\dagger)^{S+m}}{\sqrt{(S+m)!}} \frac{(a_2^\dagger)^{S-m}}{\sqrt{(S-m)!}} |0\rangle .$$

Q): how Schwinger bosons transform under spatial rotation?

Eulerian angle representation of $SU(2)$ rotation

$$R[\phi, \theta, \chi] = e^{-i\phi S_z} e^{-i\theta S_y} e^{-i\chi S_z}$$

1) rotation around z-axis at the angle of ϕ.

2) rotation around the new position of y-axis at the angle of θ

3) rotation around the new position of z-axis at the angle of χ.

Ex: prove that

$$e^{-i\chi \vec{S}\cdot\hat{e}_{z''}} e^{-i\theta \vec{S}\cdot\hat{e}_{y'}} e^{-i\phi S_z} = e^{-i\phi S_z} e^{-i\theta S_y} e^{-i\chi S_z}$$

where $\hat{e}_y'$ is the position of y-axis after rotation 1) and

$\hat{e}_{z''}$ is the position of z-axis after rotation 2).

The transformation of Schwinger bosons are defined as

$$\begin{cases} a_1'^{\dagger} = R\, a_1^{\dagger}\, R^{-1} \\ a_2'^{\dagger} = R\, a_2^{\dagger}\, R^{-1} \end{cases}$$

in order to derive $a_1'^{\dagger}$ and $a_2'^{\dagger}$. ① we first calculate $e^{-i\phi S_z}\begin{pmatrix} a_1^{\dagger} \\ a_2^{\dagger}\end{pmatrix} e^{i\phi S_z}$.

Define $f(\phi) = e^{-i\phi S_z}\begin{pmatrix} a_1^{\dagger} \\ a_2^{\dagger}\end{pmatrix} e^{i\phi S_z}$

$$\Rightarrow i\frac{d}{d\phi}f(\phi) = e^{-i\phi S_z}\begin{pmatrix} [S_z, a_1^{\dagger}] \\ [S_z, a_2^{\dagger}]\end{pmatrix} e^{i\phi S_z} = \frac{1}{2} e^{-i\phi S_z}\begin{pmatrix} a_1^{\dagger} \\ -a_2^{\dagger}\end{pmatrix} e^{i\phi S_z}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & \\ & -1\end{pmatrix} f(\phi)$$

Considering $f(\phi=0) = \begin{pmatrix} a_1^{\dagger} \\ a_2^{\dagger}\end{pmatrix}$, we can integrate it out as

or $$\boxed{e^{-i\phi S_z}\begin{pmatrix} a_1^{\dagger} \\ a_2^{\dagger}\end{pmatrix} e^{i\phi S_z} = \begin{pmatrix} e^{-i\phi/2} & \\ & e^{i\phi/2}\end{pmatrix}\begin{pmatrix} a_1^{\dagger} \\ a_2^{\dagger}\end{pmatrix}}$$

② then we calculate $g(\theta) = e^{-i\theta S_y}\begin{pmatrix} a_1^{\dagger} \\ a_2^{\dagger}\end{pmatrix} e^{i\theta S_y}$

$$\Rightarrow i\frac{d}{d\theta} g(\theta) = e^{-i\theta S_y}\begin{pmatrix} [S_y\, a_1^{\dagger}] \\ [S_y\, a_2^{\dagger}]\end{pmatrix} e^{i\theta S_y} = \begin{pmatrix} e^{-i\theta S_y}\, \frac{i a_2^{\dagger}}{2}\, e^{i\theta S_y} \\ e^{-i\theta S_y}\, \frac{-i a_1^{\dagger}}{2}\, e^{i\theta S_y}\end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} & i \\ -i & \end{pmatrix} g(\theta) \qquad \Rightarrow \begin{cases} \frac{d}{d\theta} g(\theta) = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix} g(\theta) \\ g(0) = \begin{pmatrix} a_1^{\dagger} \\ a_2^{\dagger}\end{pmatrix}\end{cases}$$

$$\Rightarrow g(\theta) = \begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ -\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} \begin{pmatrix} a_1^\dagger \\ a_2^\dagger \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} a_1'^\dagger \\ a_2'^\dagger \end{pmatrix} = R \begin{pmatrix} a_1^\dagger \\ a_2^\dagger \end{pmatrix} R^{-1} = e^{-i\phi S_z} e^{-i\theta S_y} \underbrace{e^{-i\chi S_z} \begin{pmatrix} a_1^\dagger \\ a_2^\dagger \end{pmatrix} e^{i\chi S_z}}\, e^{i\theta S_y} e^{i\phi S_z}$$

$$= e^{-i\phi S_z} e^{-i\theta S_y} \begin{bmatrix} e^{-i\chi/2} & 0 \\ 0 & e^{i\chi/2} \end{bmatrix} \begin{bmatrix} a_1^\dagger \\ a_2^\dagger \end{bmatrix} e^{i\theta S_y} e^{i\phi S_z}$$

$$= \begin{bmatrix} e^{-i\chi/2} & 0 \\ 0 & e^{i\chi/2} \end{bmatrix} e^{-i\phi S_z} \underbrace{e^{-i\theta S_y} \begin{bmatrix} a_1^\dagger \\ a_2^\dagger \end{bmatrix} e^{i\theta S_y}}\, e^{i\phi S_z}$$

$$= \begin{bmatrix} e^{-i\chi/2} & 0 \\ 0 & e^{i\chi/2} \end{bmatrix} \begin{bmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ -\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix} \begin{bmatrix} e^{-i\phi/2} & 0 \\ 0 & e^{i\phi/2} \end{bmatrix} \begin{bmatrix} a_1^\dagger \\ a_2 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\frac{\theta}{2} e^{-i\frac{\phi+\chi}{2}} & \sin\frac{\theta}{2} e^{i\frac{\phi-\chi}{2}} \\ -\sin\frac{\theta}{2} e^{i\frac{-\phi+\chi}{2}} & \cos\frac{\theta}{2} e^{i\frac{\phi+\chi}{2}} \end{bmatrix} \begin{bmatrix} a_1^\dagger \\ a_2^\dagger \end{bmatrix}$$

define $\begin{cases} u = \cos\frac{\theta}{2} e^{-i\phi/2} \\ v = \sin\frac{\theta}{2} e^{i\phi/2} \end{cases} \Rightarrow \begin{pmatrix} a_1^{\dagger\prime} \\ a_2^{\dagger\prime} \end{pmatrix} = \begin{pmatrix} u e^{-i\chi/2}, & v e^{-i\chi/2} \\ -v^* e^{i\chi/2} & u^* e^{i\chi/2} \end{pmatrix} \begin{pmatrix} a_1^\dagger \\ a_2^\dagger \end{pmatrix}$

$$\boxed{\text{or } (a_1^{\dagger\prime}, a_2^{\dagger\prime}) = (a_1^\dagger \;\; a_2^\dagger) \begin{pmatrix} u e^{-i\chi/2} & -v^* e^{i\chi/2} \\ v e^{-i\chi/2} & u^* e^{i\chi/2} \end{pmatrix}}$$

§2 spin-coherent state path integral

$$|\hat{\Omega}\rangle = R(\chi,\theta,\phi)\,|SS\rangle = e^{-iS_z\phi}\, e^{-iS_y\theta}\, e^{-iS_z\chi}\,|SS\rangle$$

$|\hat{\Omega}\rangle$ is denoted as spin coherent state and $|SS\rangle = |\Omega = \hat{z}\rangle$.

Generally. $\hat{\Omega} = (\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta)$.

what's $|\hat{\Omega}\rangle$ under the orth-normal basis?

$$|\hat{\Omega}\rangle = \frac{(a_1^{+\prime})^{S+S}}{\sqrt{2S!}}\,|SS\rangle = \left(e^{-i\chi/2}\right)^{2S}\frac{(u a_1^+ + v a_2^+)^{2S}}{\sqrt{2S!}}\,|0\rangle$$

$$= e^{-iS\chi}\sum_m \binom{2S}{S+m}\frac{u^{S+m}v^{S-m}}{\sqrt{2S!}}(a_1^+)^{S+m}(a_2^+)^{S-m}\,|0\rangle$$

$$= e^{-iS\chi}\sqrt{2S!}\sum_m \frac{u^{S+m}v^{S-m}}{\sqrt{(S+m)!}\sqrt{(S-m)!}}\,|Sm\rangle$$

Inner product

$$\langle\Omega|\Omega'\rangle = e^{iS(\chi-\chi')}\,2S!\sum_m\frac{(u^*u')^{S+m}(v^*v')^{S-m}}{(S+m)!\,(S-m)!}$$

$$= e^{iS(\chi-\chi')}\left(u^*u' + v^*v'\right)^{2S}$$

the right hand side is the inner product on the S^3-sphere with $2S$ power.

$$u^*u' + v^*v' = \cos\frac{\theta}{2}\cos\frac{\theta'}{2}\,e^{i\frac{\phi-\phi'}{2}} + \sin\frac{\theta}{2}\sin\frac{\theta'}{2}\,e^{-i\frac{\phi-\phi'}{2}}$$

$$= \cos\frac{\theta-\theta'}{2}\cos\frac{\phi-\phi'}{2} + i\sin\frac{\theta+\theta'}{2}\sin\frac{\phi-\phi'}{2}$$

$$|u^*u' + v^*v'|^2 = \cos^2\frac{\theta}{2}\cos^2\frac{\theta'}{2} + \sin^2\frac{\theta}{2}\sin^2\frac{\theta'}{2} + 2\sin\frac{\theta}{2}\sin\frac{\theta'}{2}\cos(\phi-\phi')$$

$$= \frac{1+\cos\theta}{2}\frac{1+\cos\theta'}{2} + \frac{1-\cos\theta'}{2}\frac{1-\cos\theta}{2} + \frac{1}{2}\sin\theta\sin\theta'\cos(\phi-\phi')$$

$$= \frac{1}{2}[1+\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi')]$$

$$= \frac{1}{2}(1+\hat{\Omega}\cdot\hat{\Omega}')$$

$$\Rightarrow \boxed{\langle\Omega|\Omega'\rangle = \left(\frac{1+\hat{\Omega}\cdot\hat{\Omega}'}{2}\right)^{2S} e^{iS\psi}}$$

The phas $\psi = \chi - \chi' + 2\,\mathrm{arctg}\left[\tan\frac{\phi-\phi'}{2}\,\frac{\cos\frac{\theta+\theta'}{2}}{\cos\frac{\theta-\theta'}{2}}\right]$

Remark: (u, v) canbe viewed as a coordinate on the S^2-sphere of $\hat{\Omega}$.

$$\boxed{\hat{n} = (u^*, v^*)\,\vec{\sigma}\begin{pmatrix} u \\ v \end{pmatrix}}$$

certainly if we do $\begin{pmatrix} u \\ v \end{pmatrix} \to \begin{pmatrix} u e^{-i\chi/2} \\ v e^{-i\chi/2} \end{pmatrix}$, $\hat{n}$ doesnot change. Thus χ-angle becomes a gauge degree of freedom.

This expression is called 1st Hopf map from $S^3 \longrightarrow S^2$.

resolution identity:

$$\boxed{\frac{2S+1}{4\pi}\int d\Omega\ |\hat{\Omega}\rangle\langle\hat{\Omega}| = 1}$$

Proof: $$LHS = \frac{2S+1}{4\pi}(2S)!\int d\Omega \sum_m \frac{|u|^{2S+2m}\,|v|^{2S-2m}}{(S+m)!\,(S-m)!}\,|Sm\rangle\langle Sm|$$

$$= \int\frac{d\Omega}{4\pi}\,(2S+1)!\sum_m \frac{\left(\frac{1+\cos\theta}{2}\right)^{S+m}\left(\frac{1-\cos\theta}{2}\right)^{S-m}}{(S+m)!\,(S-m)!}\,|Sm\rangle\langle Sm|$$

$$\int\frac{d\Omega}{4\pi}\left(\frac{1+\cos\theta}{2}\right)^{S+m}\left(\frac{1-\cos\theta}{2}\right)^{S-m} = \frac{1}{2}\int_{-1}^{1}dx\left(\frac{1+x}{2}\right)^{S+m}\left(\frac{1-x}{2}\right)^{S-m}$$

set $y=\frac{x+1}{2}$ $\Rightarrow$ $\Rightarrow$ $$= \int_0^1 dy\ y^{S+m}(1-y)^{S-m}$$

$$= \frac{(S+m)!\,(S-m)!}{(2S+1)!}$$

$$\Rightarrow LHS = \sum_m |Sm\rangle\langle Sm| = 1.$$

§ path integral Rep of partition function (a single spin)

$$Z = tr[e^{-\beta \hat{H}}] = \lim_{N_\epsilon \to \infty} tr \prod_{n=0}^{N_\epsilon - 1} (1 - \epsilon \hat{H}(\tau_n))$$

$\hat{H}$ is Hamitonian for a single spin, say $\hat{H} = -\vec{B}\cdot\vec{S}$

Insert Resolution indentity $\Rightarrow$

$$Z = \lim_{N_\epsilon \to \infty} \int \prod_{\tau} d\Omega(\tau) \prod_{\tau=\epsilon}^{\beta} \langle \Omega(\tau) | 1 - \epsilon \hat{H}(\tau) | \Omega(\tau-\epsilon) \rangle$$

$$\langle \Omega(\tau) | \Omega(\tau-\epsilon) \rangle \xrightarrow[\text{linear order}]{\text{keep } \epsilon\text{'s}} e^{iS\psi} \leftarrow \left(\frac{1 + \hat{\Omega}\cdot\hat{\Omega}'}{2} \right)^{2S} \approx 1$$

and $\psi \approx \epsilon \dot{\chi} + \dot{\phi} \epsilon \cos\theta$

$$\Rightarrow \langle \Omega(\tau) | \Omega(\tau-\epsilon) \rangle \simeq \exp[i \epsilon S (\dot{\chi} + \dot{\phi} \cos\theta)]$$

as for $\langle \Omega(\tau) | \hat{H}(\tau) | \Omega(\tau-\epsilon) \rangle$, we only need to keep to zeroth order of ϵ, thus $\rightarrow$ $\langle \Omega(\tau) | \hat{H}(\tau) | \Omega(\tau) \rangle = H(\hat{\Omega}_i)$

because $(\hat{\Omega}\cdot\vec{S}) |\hat{\Omega}\rangle = S |\hat{\Omega}\rangle$

$$\Rightarrow \quad Z = \oint D\hat{\Omega}(\tau) \exp\Big[\sum_{n=0}^{N\epsilon-1} i \epsilon S [(\dot{\chi} + \dot{\phi}\cos\theta) - \sum_{n=0}^{N\epsilon-1} \epsilon H(\hat{\Omega}_i)\Big]$$

we can organize as

$$\boxed{\begin{aligned} Z &= \oint D\hat{\Omega}(\tau) \exp[-S(\hat{\Omega})], \text{ with} \\ S[\hat{\Omega}] &= iS\, \omega(\Omega) + \int_0^\beta d\tau\, H(\hat{\Omega}(\tau)) \end{aligned}}$$

In order to have the geometric expression of $\omega(\Omega) = \int d\tau\, [-\dot{\chi} - \dot{\phi}\cos\theta]$, we need to choose the gauge of χ. Physically, θ and ϕ are the direction of $\hat{\Omega}$, and we have the boundry condition $\hat{\Omega}(\beta) = \hat{\Omega}(0)$. We would also let $\begin{pmatrix} u \\ v \end{pmatrix} e^{-i\chi/2} = \begin{pmatrix} \cos\frac{\theta}{2}\, e^{-i(\phi+\chi)/2} \\ \sin\frac{\theta}{2}\, e^{i(\phi-\chi)/2} \end{pmatrix}$ also periodic.

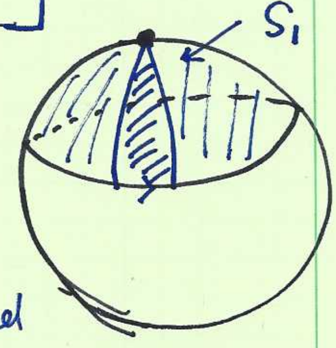
when the loop of $\hat{\Omega}$ enclose the north pole, we have $\varphi \to \varphi \pm 2\pi$, the factor $e^{i\phi/2}$ causes problem. (It changes sign). On the other hand, the choice of north pole is arbitany. We would like a geometric expression of $\omega(\Omega)$, which only depends on the loop of $\hat{\Omega}(\tau)$, but not on whether the north pole is enclosed or not.

(If we insist to use $\chi = 0$, then we need to take into acount the jump of ϕ at the boundary).

Then we can choose $\chi = -\phi$, then we have

$$\boxed{\omega(\Omega) = \oint d\tau \; \dot{\phi}(1-\cos\theta) = \oint d\phi \,(1-\cos\theta)} \leftarrow S_1$$

This is nothing but the area enclosed by the trajectory.



Remark:

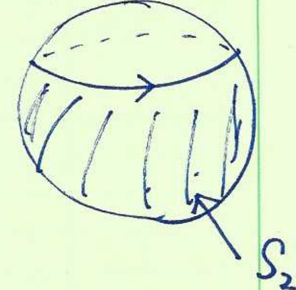
Then a question appear: on the sphere, a closed loop could be interpret to enclose 2 areas. How do we choose?

The other choice is equivalent to the choice of $\chi = \phi$.

Then

$$\omega(\Omega) = -\oint d\tau \; \dot{\phi}(1+\cos\theta)$$

$$= -S_2$$



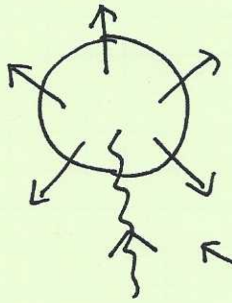
we have $S_1 + S_2 = 4\pi$, the spin value is intege or half integer, thus these two different choices do not lead trouble.

$$\boxed{\omega(\Omega) \text{ is the Berry phase} \leftarrow \text{area enclosed by the loop.}}$$

$$\omega = \int_0^\beta d\tau \; \vec{A}(\Omega)\, d\hat{\Omega} \quad \text{where} \quad (\nabla \times \vec{A}) \cdot \hat{\Omega} = 1$$

$\vec{A}$ is the vector potential for a magnetic monopole.

A standard choice is $\vec{A} = \frac{1-\cos\theta}{\sin\theta}\,\hat{e}_\phi = \tan\frac{\theta}{2}\,\hat{e}_\phi$



§ Equation of motion and large-S expansion

When in the real-time, we have

$$G(\Omega_t,\Omega_0;t) = \langle \Omega_t |\, T \exp\int_0^t dt' \,(-i H(t'))\, | \Omega_0 \rangle \leftarrow \tau \to it$$

$$= \int_{\Omega_0}^{\Omega_t} D\Omega(t')\, \exp[\, i S(\hat{\Omega})\,]$$ where $S[\hat{\Omega}]$ is the real time action

$$\boxed{S[\hat{\Omega}] = \int_0^t dt' \left[S\, \vec{A}\cdot\dot{\hat{\Omega}} - H(\Omega(t')) \right].}$$

We will find the saddle point equation

$$\frac{\delta}{\delta\hat{\Omega}} S[\hat{\Omega}] \Big|_{\Omega^{cl}} = 0,$$ where the boundary condition

$\hat{\Omega}^{cl}(0) = \hat{\Omega}_0$ and $\hat{\Omega}^{cl}(t) = \hat{\Omega}_t$.

$$\delta\left[\int_0^t \vec{A}\cdot\dot{\hat{\Omega}}\,dt'\right] = \int_0^t dt' \; \frac{\delta A^\alpha}{\delta\Omega^\beta}\,\delta\Omega^\beta\,\dot{\hat{\Omega}}^\alpha + A^\alpha \frac{d}{dt'}\delta\hat{\Omega}^\alpha$$

$$+\left(\frac{\partial A^\alpha}{\partial\Omega^\beta}\,\dot{\Omega}^\beta\,\delta\hat{\Omega}^\alpha - \frac{\partial A^\alpha}{\partial\Omega^\beta}\,\dot{\Omega}^\beta\,\delta\hat{\Omega}^\alpha\right)$$

this term is zero to rearrange the first line

$$= \int_0^t dt' \; \frac{\partial A^\alpha}{\delta\Omega^\beta}\left[\dot{\Omega}^\alpha\,\delta\Omega^\beta - \dot{\Omega}^\beta\,\delta\Omega^\alpha\right) + \int_0^t dt' \; \underbrace{A^\alpha \frac{d}{dt'}\delta\Omega^\alpha + \frac{dA^\alpha}{dt'}\,\delta\Omega^\alpha}_{\frac{d}{dt'}(\vec{A}\cdot\delta\hat{\Omega})}$$

The last term $=0$ due to the boundary condition.

The first term $= \displaystyle\int_0^t dt' \; \frac{\partial A^\alpha}{\partial\Omega^\beta}\,[\dot{\Omega}^{\alpha'}\delta\Omega^{\beta'}]\,[\delta_{\alpha\alpha'}\delta_{\beta\beta'} - \delta_{\alpha'\beta}\delta_{\alpha\beta'}]$

$$= \int_0^t dt' \; \frac{\partial A^\alpha}{\partial\Omega^\beta}\,[\dot{\Omega}^{\alpha'}\delta\Omega^{\beta'}]\;\epsilon^{\gamma\alpha\beta}\,\epsilon^{\gamma\alpha'\beta'}$$

$$= \int_0^t dt' \; \underbrace{\epsilon^{\gamma\alpha\beta}\frac{\partial A^\alpha}{\partial\Omega^\beta}}_{\text{field streigh} = \Omega^\gamma} \;\epsilon^{\gamma\alpha'\beta'}\,[\dot{\Omega}^{\alpha'}\delta\Omega^{\beta'}]$$

$$\Rightarrow \int_0^t dt' \; \hat{\Omega}\cdot[\dot{\hat{\Omega}}\times\delta\hat{\Omega}] = \int_0^t dt' \; \delta\hat{\Omega}\cdot(\hat{\Omega}\times\dot{\hat{\Omega}})$$

then $\displaystyle \delta S[\hat{\Omega}] = \int_0^t dt' \left[(\hat{\Omega}\times\dot{\hat{\Omega}})S - \frac{\partial H}{\partial\Omega}\right]\cdot\delta\Omega$

$$\Rightarrow \boxed{S\,\hat{\Omega}\times\dot{\hat{\Omega}} = \frac{\partial H}{\partial\Omega}[\hat{\Omega}]}$$

$$\left.\begin{aligned} &S\,\hat{\Omega} \times [\hat{\Omega} \times \dot{\hat{\Omega}}] = \hat{\Omega} \times \frac{\partial H}{\partial \Omega}(\hat{\Omega}) \\ &\to (\hat{\Omega} \cdot \dot{\hat{\Omega}})\,\hat{\Omega} - (\Omega \cdot \Omega)\,\dot{\hat{\Omega}} = -\dot{\hat{\Omega}} \end{aligned}\right\} \Rightarrow S\dot{\hat{\Omega}} = \hat{\Omega} \times \left(-\frac{\partial H}{\partial \hat{\Omega}}\right)$$

if $H = -\vec{B} \cdot \vec{S} = -\vec{B} \cdot \hat{\Omega}\, S$

$$\Rightarrow \boxed{\dot{\hat{\Omega}} = \hat{\Omega} \times \vec{B}}$$

Lamor procession