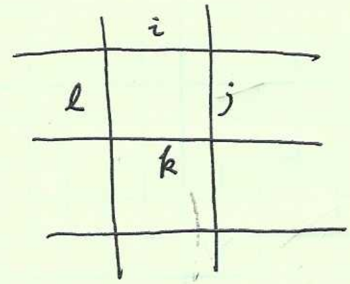
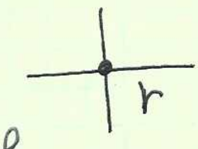
# $Z_2$ gauge theory




Define Ising variables on each bonds

$$Z(K) = \sum_S \exp\left[K \sum_{\square} S_i S_j S_k S_l\right]$$

Gauge symmetry: $S_i \to -S_i$ for all bonds attached to a site r




(It flips the sign of two bonds of the plaquettes taking "r" as its corner). This is a local symmetry. Or for each site, we can define such a symmetry ($Z_2$-gauge sym)

$$S_r(S_i) = -S_i \quad \text{for } i \in \text{Bond}(r),$$

Only gauge invariant quantity can have nonzero expectation value — say, the product of spin around a closed loop, (Wilson loop) can develop a nonzero expection value.

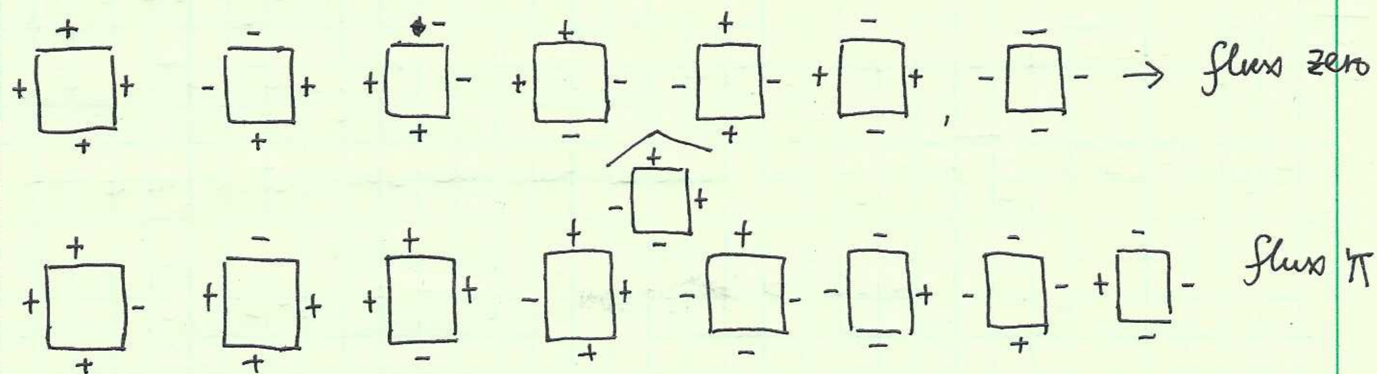
Wegner

(*) Elizur theorem — gauge symmetry cannot be broken.

Explanation: for a global symmetry, it can be spontaneously broken in the thermodynamic limit. The energy barrier separating different configurations scale with the system size. Hence ergodicity is violated. But gauge symmetry is local, the energy barrier is finite. Different configurations connected by gauge symmetry have ergodicity, which means the gauge symmetry cannot be broken.

⊛ Gauge fixing: We only consider gauge invariant (GI) observables.

Gauge theory provides an reduant description: Different configurations related by gauge thransformations yield the same result. We can classify configurations into different gauge inequivalent ~~cat~~ classes. For example




The 16 configuratutions $\longrightarrow$ only two non-equivalent classes!

$$\langle O_{GI} \rangle = \frac{\sum_s O_{GI}\, e^{-S(s)}}{\sum_s e^{-S(s)}} = \frac{\sum_{GI} g\, O_{GI}\, e^{-S(s)}}{\sum_{GI} g\, e^{-S(s)}}$$

$\Sigma_{GI}$ means sum over one representative configuration in each gauge invariant class. $g$ is the number of configurations of each class. This is called gauge fixing.

(*) High ~~temperature~~ limit

$$Z = \sum_S e^{K \sum_{\square} S_i S_j S_k S_l} = \sum_S \prod_{\square} \cosh K \left[ 1 + \tanh K \, S_i S_j S_k S_l \right]$$

There are N plaquette $\to$ $(\cosh K)^N$, $\sum_S$ sum over $2^{2N}$ $\overbrace{\text{configuration}}^{\text{bond}}$

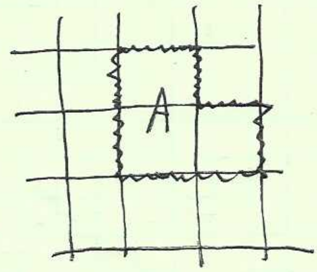
Hence the leading order $(2^2 \cosh K)^N$. We have to pair all the Ising variables on the link. The only possibility is to cover all the plaquettes in the plane with periodical boundary condition $\Rightarrow$

$$Z = (4 \cosh K)^N \left[ 1 + (\tanh K)^N \right]$$

$\leftarrow$ N is the # of plaquette of Ising gauge model.

This is the same in the 1d Ising model, with length N.

$\boxed{\text{Area law:}}$ Consider a gauge invariant operator $O_\Gamma$, which a product of operators around a loop. The area enclosed by the loop is A.




$$\langle O_\Gamma \rangle = \left\langle \prod_{i \in \Gamma} S_i \right\rangle = \frac{(\tan K)^A}{1 + (\tan K)^N} \xrightarrow{K \to 0} \propto K^A$$

we need to choose the plaquettes forming the area A. The boundary bonds are balanced by the $O_\Gamma$.

Hence the Wegner loop decays with an area law.

In 2D Ising gauge model, we can use the flux of each plaqutte as an independent degree of freedom. (up to a constraint that the product of all fluxes = 1). Then partition function $\to (2\cosh K)^N$.

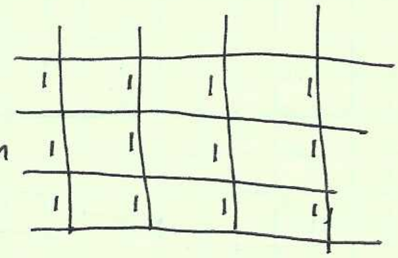
$$\langle O_\Gamma \rangle = \frac{\sum_{p(\square)} \left( \prod_{\square \text{ inside } \Gamma} p(\square) \right) \prod_\square e^{Kp\square}}{\sum_{p(\square)} \prod_\square e^{Kp(\square)}}$$

$$= \frac{\prod_{\square \text{ outsider}} (e^K + e^{-K}) \prod_{\square \text{ insider } \Gamma} (e^K - e^{-K})}{\prod_\square (e^K + e^{-K})} = (\tan K)^A \sim \text{up to boundary effect.}$$

- Gauge fixing — temporal gauge.

- We choose $S_i = +1$ for all vertical bond

For open boundary condition, for any configuration, we can apply ~~gauge tranformation~~ to arrive at the temporal gauge.




But for periodical bondary condition, it's not always applicable.

Then in the temproal gauge

$$S = K \sum_{i=1}^{M} \sum_{j=1}^{N-1} S_{ij} S_{i,j+1}$$

$\to$ decoupled 1D Ising chains, hence, there's only a single phase.

§ 3D Ising Gauge theory



$$Z = \sum_{S} \exp\left[K \sum_{\square} S_i S_j S_k S_e\right] = \sum_{S} \prod_{\square} \cosh K \left[1 + \tan k\, S_i S_j S_k S_e\right]$$

Now if the plaquttes form closed surfaces, then the average of the product of $\prod_{\square} S_i S_j S_k S_e$ won't vanish. The lowest order contribution comes from the six surfaces surrounding a cube — 6th order. $\Rightarrow$

$$Z = (\cosh K)^{3N} \cdot 2^{3N} \left[1 + N (\tanh k)^6 + \cdots \right]$$

N sites — 3N bonds, 3N faces

$$\Rightarrow \frac{Z}{(2\cosh K)^{3N}} = \sum_{A} g(A) (\tanh k)^A,$$

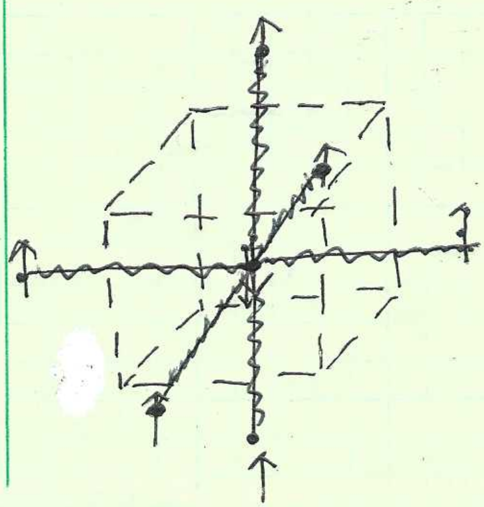
where $g(A)$ is # of closed surfaces with the Area.

If we consider a 3d Ising model, and do a low T expansion

$$Z = \sum_{\langle ij \rangle} e^{K^* \sigma_i \sigma_j}$$

Consider the fully polarized phase

$$2 e^{3NK^*} \cdot \left(1 + \#\text{broken bonds}\; e^{-2K^*} \cdots\right)$$



we can cut the bisector planes of broken bonds, which form closed surfaces. Hence the # of broken bonds maps to the area of domain wall surfaces.

$$\Rightarrow \quad \frac{Z(K)}{(2\cosh K)^{3N}} = \frac{Z(K^*)}{2(e^{3NK})^*}$$

if we identify $\tanh K = e^{-2K^*}$, i.e. the duality realation

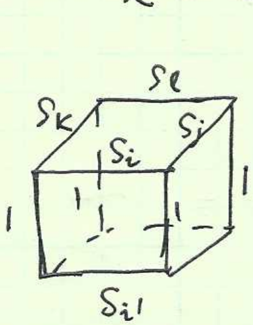
$$\sinh 2K \sinh 2K^* = 1.$$

Since we know 3d Ising model has two different phases: the low T ordered phase and the high T disordered phase, what we can conculde for the 3d Ising gauge model? It's reflected in $\langle O_\Gamma \rangle$, it decays expontially at small K (high T), of the A of $\Gamma$ and decays expontionally at the perimeter of $\Gamma$ at largevalue of K (low T).

(*) Hamiltonian treatment $d=3$ (tempural gauge)

Consider an anisotropic 3d Ising gauge model, with $K_\tau$ along the vertical direction, and K along the x-y direction. In the tempural gauge

$$Z = e^S \quad \text{with} \quad S = K_\tau \sum S(i)\, S(i') + K \sum S(i) S(j) S(l) S(k)$$




The transfermatrix is defined for all the bonds on the 2D plane, and the temproal term can be captured by

$$e^{K_\tau^* \sigma_x(i)}$$

and that in the x-y plane

$$e^{K \sum \sigma_z(i)\sigma_z(j)\sigma_z(k)\sigma_z(k)}$$

we conside the limit that $K = \tau \to 0$, $K_\tau^* = \lambda\tau = \tanh^{-1} e^{-2K_\tau}, \to 0$.

Then we arrive at a $\tau$- continous Hamiltonian $Z = \mathrm{Tr}[(e^{-\tau H})^{N_{vertical}}]$

$$H = -\sum_{\square} \sigma_3(i)\sigma_3(j)\sigma_3(k)\sigma_3(l) - \lambda \sum_{bonds} \sigma_1(i)$$

where $\lambda \to 0$, the low ~~temperture~~ phase, and $\lambda \to \infty$ high T phase.

Since the model is not self-dual, the precise critical ~~temperature~~ is not easy to solve.

⊛ Conside the limit $\lambda \to \infty$, then $\lambda$-term dominates. Define $O_\Gamma$ with $\Gamma$ loops in the xy-plane (equal time)

$$\langle O_\Gamma \rangle = \langle G | \prod_{i \in \Gamma} \sigma_3(i) | G \rangle .$$

At $\lambda = \infty$, since each site is an eigenstate of $\sigma_1 \Rightarrow \langle \sigma_3 \rangle = 0$, and $\langle O_\Gamma \rangle$ vanishes. At $\lambda$ large but finit, we can treat $H' = \frac{-1}{\lambda} \sum_{\square} \sigma_3(i)\sigma_3(j)\sigma_3(k)\sigma_3(l)$ as perturbative Hamiltonian.

$|G\rangle_0$ : all the bonds carry $\sigma_1$ eigen value $+1$. $\sigma_3$ flips the $\sigma_1$'s eigenvalue from $+1 \to -1$. Each term in $H'$ flips the signs of $\underbrace{\text{a plagutte}}_{\sigma_1\text{'s around}}$. We need apply perturbation $H'$

A times to flip all the signs of plagutte bonds inside $\Gamma$. Then the internal bonds remain unchanged, and the bonds along the boundary are flipped. This is combined with $O_\Gamma$ to reach a nonzero expectation value.

Hence $\langle G | O_\Gamma | G \rangle \simeq \left(\frac{1}{\lambda}\right)^A \simeq e^{-A \ln \lambda}$.

We arrive at an area law.

(*) The low-~~temperature~~ phase $(\lambda \to 0)$

If $\lambda = 0^+$, the ground state $|G\rangle_0$ is an equal superposition of all ~~states~~ states with flux $= 1$. We ~~start~~ start with all bonds $+1$ and perform gauge transformations. $\longrightarrow$ gauge invariance.

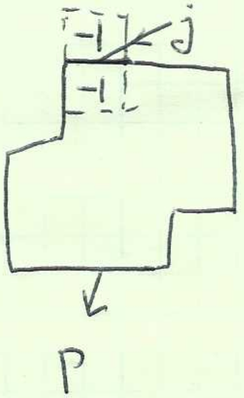
To linear order of $\lambda$, only one bond is fliped, which cause two adjacent plaquttes unhappy: fluxes $+1 \to -1$, The energy cost is 4.

$$|G\rangle = |G\rangle_0 + \frac{\lambda}{4} \sum_{\text{bonds}} \sigma_1(i) |G\rangle_0$$

$$= |G\rangle_0 + \frac{\lambda}{4} \sum_{j \in \Gamma} |\text{bond } j \text{ flipped}\rangle + \frac{\lambda}{4} \sum_{j \notin \Gamma} |\text{bond } j \text{ flipped}\rangle$$

(the flipped bond "$i$" can belong to $\Gamma$ or not).

Then
$$\prod_{i \in \Gamma} \sigma_3(i) |G\rangle = |G\rangle_0 - \frac{\lambda}{4} \sum_{j \in \Gamma} |\text{bond } j \text{ flipped}\rangle + \frac{\lambda}{4} \sum_{j \notin \Gamma} |\text{bond } j \text{ flipped}\rangle$$




$$\Rightarrow \prod_{i\in P} \sigma_3(i)\ |\text{bond } j \text{ flipped}\rangle = \mp 1\ |\text{bond } j \text{ flipped}\rangle$$

$\mp$: $-$ if $j \in P$, $+$ if $j \notin P$

$$\Rightarrow \langle G|G\rangle = 1 + \frac{\lambda^2 N}{16}$$

$$\frac{\langle G|\prod_{i\in P}\sigma_3|G\rangle}{\langle G|G\rangle} = \frac{1+\frac{\lambda^2}{16}(N-L)-\frac{\lambda^2}{16}L}{1+\frac{\lambda^2 N}{16}} \approx 1-\frac{\lambda^2 L}{8}$$

$$\approx e^{-\frac{\lambda^2 L}{8}}$$

Only the flipping along the loop suppresses the correlation.

Hence we can divide the phases into an area exponetionally decay phase and a perimeter-exponetionally decay phase.