Warm up spin 1/2 Hubbard U, V model

$$H = -t \sum_{\langle ij \rangle} C^{+}_{i\sigma} C_{j\sigma} + U \sum_{\langle i \rangle} n_{i\uparrow} n_{i\downarrow} + V \sum_{\langle i, i+1 \rangle} n_i n_{i+1}$$

useful identity : define $N^{+} = \psi^{+}_{R\sigma} \psi_{L\sigma}$, $\vec{N}^{+} = \psi^{+}_{R\alpha} \left(\frac{\vec{\sigma}}{2}\right)_{\alpha\beta} \psi_{L\beta}$

$$N^{+} N = -\frac{1}{2} J_R \cdot J_L - 2 \vec{J}_R \cdot \vec{J}_L \ , \qquad \vec{N}^{+} N = -\frac{3}{8} J_R J_L + \frac{1}{2} \vec{J}_R \cdot \vec{J}_L$$

$$\therefore \ n_{i\uparrow} n_{i\downarrow} = \frac{1}{2} (n_{i\uparrow} + n_{i\downarrow})^2 - n_{i\uparrow} - n_{i\downarrow} = \frac{1}{2} [J_R + J_L + N^{+} e^{-2ik_F x} + N e^{2ik_F x}]^2$$

$$\longrightarrow \ J_R \cdot J_L + N^{+} N = \boxed{\frac{1}{2} J_R J_L - 2 \vec{J}_R \cdot \vec{J}_L \ , \quad + e^{-i4k_F x} (\psi^{+}_{R\uparrow} \psi^{+}_{R\downarrow} \psi_{L\downarrow} \psi_{L\uparrow}) + h.c.}$$

$$+ \frac{1}{2} (N^{+} N^{+} e^{-4ik_F x} + h.c.)$$

where $J_{R,L} = \psi^{+}_{R\sigma} \psi_{L\sigma}$

$\vec{J}_{R,L} = \psi^{+}_{R\alpha} \left(\frac{\vec{\sigma}}{2}\right)_{\alpha\beta} \psi_{L\beta}$

$$n(x) n(x+a) = [J_L(x) + J_R(x) + e^{-i2k_F x} N^{+}(x) + e^{i2k_F x} N(x)] [J_L(x) + J_R(x) + e^{-2ik_F x} e^{-2ik_F a} N^{+}(x) + e^{2ik_F x} e^{2ik_F a} N(x)]$$

$$\sim 2 J_L(x) J_R(x) + (e^{2ik_F a} + e^{-2ik_F a}) N^{+}(x) N(x) + \{ e^{-i4k_F x} e^{-2ik_F a} N^{+}(x) N^{+}(x) + h.c. \}$$

$$= 2 J_L(x) J_R(x) + 2\cos 2k_F a \left[-\frac{J_R J_L}{2} - 2 \vec{J}_R \cdot \vec{J}_L \right] + 2 \left[e^{-i4k_F x} e^{-2ik_F a} \psi^{+}_{R\uparrow} \psi^{+}_{R\downarrow} \psi_{L\downarrow} \psi_{L\uparrow} + h.c \right]$$

So only including non-chiral interaction:

$$H_0 = \frac{\pi}{2} v_F \int dx \, (J_L^2 + J_R^2) + \frac{2\pi}{3} v_F \int dx \, [\vec{J}_R^{\,2} + \vec{J}_L^{\,2}]$$

$$H_{int} = \left[\frac{U}{2} + (2 - \cos 2k_F) V \right] J_L J_R + (-2U - 4V \cos 2k_F) \vec{J}_R \cdot \vec{J}_L$$

$$+ \left\{ [U + 2V e^{-2ik_F a}] \psi^{+}_{R\uparrow} \psi^{+}_{R\downarrow} \psi_{L\downarrow} \psi_{L\uparrow} + h.c. \right\}$$

$$J_L = \sqrt{\frac{2}{\pi}}\, \partial_x \phi_{c,L} \quad , \quad J_R = \sqrt{\frac{2}{\pi}}\, \partial_x \phi_{c,R}$$

$$J_L^2 + J_R^2 = \frac{2}{\pi}\left\{ [\partial_x \phi_{cL}]^2 + (\partial_x \phi_{c,R})^2 \right\} = \frac{1}{\pi}\left[(\partial_x \phi_c)^2 + (\partial_x \theta_c)^2 \right]$$

$$J_L \cdot J_R = \frac{2}{\pi}\left[\partial_x \phi_{cL} \cdot \partial_x \phi_{cR} \right] = \frac{1}{2\pi}\left[(\partial_x \phi_c)^2 - (\partial_x \theta_c)^2 \right]$$

$$\rightarrow \quad H_c = \frac{1}{2}(v_F)\left[(\partial_x \phi_c)^2 + (\partial_x \theta_c)^2 \right] + \frac{1}{2\pi}\left[\frac{U}{2} + (2 - \cos 2k_F)V \right]\left[(\partial_x \phi_c)^2 - (\partial_x \theta_c)^2 \right]$$

$$- \frac{(U - 2V)}{2(\pi a)^2} \cancel{8} \cos\left(\sqrt{8\pi}\, \phi_c - (4k_F - G)x \right)$$

$$\psi^+_{R\uparrow} \psi^+_{R\downarrow} \psi_{L\downarrow} \psi_{L\uparrow} = \frac{1}{(2\pi a)^2}\, e^{-i\sqrt{4\pi}\,\phi_{R\uparrow}}\, e^{-i\sqrt{4\pi}\,\phi_{L\uparrow}}\, e^{-i\sqrt{4\pi}\,\phi_{R\downarrow}}\, e^{-i\sqrt{4\pi}\,\phi_{L\downarrow}}$$

$$= \frac{1}{(2\pi a)^2}\, e^{-i\sqrt{4\pi}\,\phi_\uparrow}\, e^{-\frac{4\pi}{2}[\phi_{R\uparrow}, \phi_{L\uparrow}]}\, e^{-i\sqrt{4\pi}\,\phi_\downarrow}\, e^{-\frac{4\pi}{2}[\phi_{R\downarrow}, \phi_{L\downarrow}]}$$

$$[\phi_R(x), \phi_L(x')] = i/4$$

$$= \frac{-1}{(2\pi a)^2}\, e^{-i\sqrt{8\pi}\,\phi_c}$$

i.e. $$H_c = \frac{v_c}{2}\left[\frac{1}{K_c}(\partial_x \phi_c)^2 + K_c (\partial_x \theta_c)^2 \right] - \frac{g_c}{2(\pi a)^2} \cos\left(\sqrt{8\pi}\,\phi_c - (4k_F - G)x \right)$$

where $$v_c = \sqrt{v_F^2 - \left[\frac{1}{\pi}\left(\frac{U}{2} + 3V \right) \right]^2}\,, \qquad K_c = \sqrt{\frac{v_F - \frac{1}{\pi}\left[\frac{U}{2} + 3V \right]}{v_F + \frac{1}{\pi}\left[\frac{U}{2} + 3V \right]}}$$

$$g_c = (U - 2V)$$

$$J_R^z = \sqrt{\frac{1}{2\pi}}\,\partial_x \phi_{sR} \qquad J_L^z = \sqrt{\frac{1}{2\pi}}\,\partial_x \phi_{s,L}$$

$$H_s = \frac{1}{2}(v_F)\left[(\partial_x \phi_s)^2 + (\partial_x \theta_s)^2\right] - (2U - 4V)\frac{1}{8\pi}\left[(\partial_x \phi_s)^2 - (\partial_x \theta_s)^2\right]$$

$$+ \frac{(U-2V)}{2(\pi a)^2}\cos\sqrt{8\pi}\,\phi_s$$

$$\frac{J_R^+ J_L^- + J_R^- J_L^+}{2} = \frac{-1}{2}\left[\psi^\dagger_{R\uparrow}\psi^\dagger_{L\downarrow}\psi_{R\downarrow}\psi_{L\uparrow} + h.c.\right] = \frac{-1}{2}\frac{1}{(2\pi a)^2}\left[e^{-i\sqrt{4\pi}\phi_{R\uparrow}} e^{-i\sqrt{4\pi}\phi_{L\uparrow}} e^{i\sqrt{4\pi}\phi_{L\downarrow}} e^{i\sqrt{4\pi}\phi_{R\downarrow}} + h.c.\right]$$

$$= \frac{-1}{2}\frac{1}{(2\pi a)^2}\left[e^{-i\sqrt{4\pi}\phi_\uparrow} e^{-\frac{4\pi}{2}\frac{i}{4}} \cdot e^{i\sqrt{4\pi}\phi_\downarrow} e^{\frac{-4\pi}{2}\frac{-i}{4}} + h.c.\right] = \frac{-1}{(2\pi a)^2}\cos\sqrt{8\pi}\,\phi_s$$

$$\boxed{H_s = \frac{v_s}{2}\left[\frac{1}{K_s}(\partial_x \phi_s)^2 + K_s(\partial_x \phi_s)^2\right] + \frac{g_s}{2(\pi a)^2}\cos\sqrt{8\pi}\,\phi_s}$$

$$v_s = \sqrt{(v_F)^2 - \left(\frac{U-2V}{2\pi}\right)^2} \qquad K_s = \sqrt{\frac{v_F + \frac{U-2V}{2\pi}}{v_F - \frac{U-2V}{2\pi}}}$$

$$g_s = U - 2V$$

g_c	g_s	$\langle\phi_c\rangle$	ϕ_s	
$+\infty$	0	0	/	Mott phase
	$-\infty$	0	0	O_{BW} :
0	0	/	/	Luttinger liquid
	$-\infty$	/	0	Luther - Emery spin gap phase
$-\infty$	0	$\sqrt{\pi/8}$	/	Mott phase
	$-\infty$	$\sqrt{\pi/8}$	0	O_{CDW}

order operator :

$$\mathcal{O}_{CDW} = \frac{2}{\pi a} \sin(\sqrt{2\pi}\phi_c - \delta\pi x) \cos\sqrt{2\pi}\phi_s$$

$$\mathcal{O}_{BW} = \frac{2}{\pi a} \cos(\sqrt{2\pi}\phi_c - \delta\pi x - \frac{\pi}{2}\delta) \cos\sqrt{2\pi}\phi_s$$

$$\mathcal{O}_{SDW}\begin{pmatrix} z \\ x \\ y \end{pmatrix} = \frac{1}{\pi a} \cos(\sqrt{2\pi}\phi_c + \delta\pi x) \begin{pmatrix} \sin\sqrt{2\pi}\phi_s \\ -i\eta_\uparrow\eta_\downarrow \sin\sqrt{2\pi}\theta_s \\ -i\eta_\uparrow\eta_\downarrow \cos\sqrt{2\pi}\theta_s \end{pmatrix}$$

$$\Delta = \frac{\eta_\uparrow\eta_\downarrow}{\pi a} \frac{1}{\sqrt{2}} e^{i\sqrt{2\pi}\theta_c} \cos\sqrt{2\pi}\phi_s$$

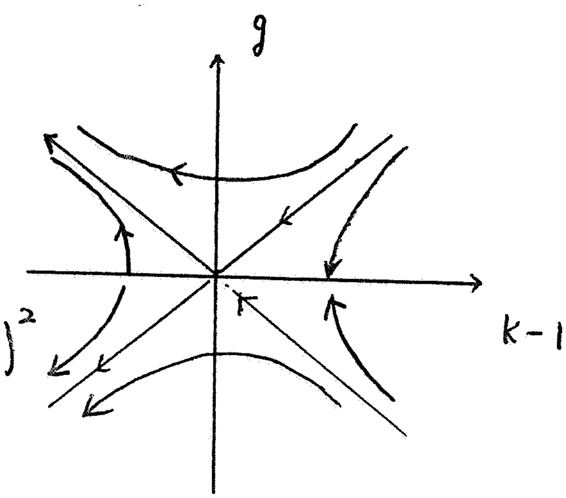
$$\Delta_{10} = \frac{i\eta_\uparrow\eta_\downarrow}{\pi a} \frac{1}{\sqrt{2}} e^{i\sqrt{2\pi}\theta_c} \sin\sqrt{2\pi}\phi_s$$

$$x \quad \frac{1}{\pi a} \frac{1}{\sqrt{2}} e^{i\sqrt{2\pi}\theta_c} \cos\sqrt{2\pi}\theta_s$$

$$y \quad \frac{1}{\pi a} \frac{1}{\sqrt{2}} e^{i\sqrt{2\pi}\theta_c} \sin\sqrt{2\pi}\theta_s$$

RG

$$\begin{cases} \dfrac{dg}{d\ln\lambda} = (2-2K)\,g \\[2ex] \dfrac{dK}{d\ln\lambda} = -\dfrac{K^2}{2\pi^2}\, g^2 \end{cases}$$



Correct to second order

$$\frac{d(K-1)^2}{d\ln\lambda} = -\frac{g^2}{\pi^2}(K-1) = \frac{1}{2\pi^2}\, g\,dg = d\left(\frac{g}{2\pi}\right)^2$$

$\Rightarrow$ seperaix $\quad K-1 = \pm \dfrac{g}{2\pi}$

we can also use some simple method: $\boxed{\text{for spin } 1/2 \text{ case}}$

Let's use spin 1/2 system as an example $-\psi^+_{R\alpha}\psi_{R\beta}\psi^+_{L\beta}\psi_{L\alpha} = N^+N$

$$N^+N = a\,\psi^+_{R\alpha}\psi_{R\alpha}\,\psi^+_{L\beta}\psi_{L\beta} + \frac{b}{4}\left(\psi^+_{R\alpha}\vec{\sigma}_{\alpha\beta}\psi_{R\beta}\right)\left(\psi^+_{L\gamma}\vec{\sigma}_{\gamma\delta}\psi_{L\delta}\right)$$

Choose $\alpha=\beta=\uparrow \Rightarrow \quad -1 = a + \frac{b}{4}$

left hand side $-\psi^+_{R\uparrow}\psi_{L\uparrow}\psi^+_{L\uparrow}\psi_{L\uparrow}$

choos $\alpha=\uparrow,\ \beta=\downarrow$, LHS: $-\psi^+_{R\uparrow}\psi_{R\downarrow}\psi^+_{L\downarrow}\psi_{L\uparrow} \Rightarrow -1 = 0 + \frac{b}{4}(1+1)$

$\Rightarrow \quad a=-1/2,\ b=-2 \quad \Rightarrow \ :N^+N: = -\frac{J_R J_L}{2} - 2\vec{J}_R\vec{J}_L$ (in the sense of normal order).

Let's check $\vec{N}^+\vec{N} = \frac{-1}{4} :\psi^+_{R\alpha}\psi_{R\delta}::\psi^+_{L\gamma}\psi_{L\beta}: (\sigma^a)_{\alpha\beta}(\sigma^a)_{\gamma\delta}$

$$= a\,\psi^+_{R\alpha}\psi_{R\alpha}\,\psi^+_{L\beta}\psi_{L\beta} + \frac{b}{4}\,\psi^+_{R\alpha}\sigma^a_{\alpha\beta}\psi_{R\beta}\,\psi^+_{L\gamma}\sigma_{\gamma\delta}\psi_{L\sigma}$$

Choose $\alpha=\beta=\gamma=\delta=\uparrow$

$LHS = \frac{-1}{4}\psi^+_{R\uparrow}\psi_{R\uparrow}\psi^+_{L\uparrow}\psi_{L\uparrow} \quad \Rightarrow \quad \frac{-1}{4} = a + \frac{b}{4}$

Choose $\alpha=\delta=\uparrow,\quad \gamma=\beta=\downarrow$

LHS: $\frac{-1}{4}\psi^+_{R\uparrow}\psi_{R\uparrow}\psi^+_{L\downarrow}\psi_{L\downarrow}\left(\sigma^x_{\uparrow\downarrow}\sigma^x_{\downarrow\uparrow} + \sigma^y_{\uparrow\downarrow}\sigma^y_{\downarrow\uparrow}\right) = -\frac{1}{2}\psi^+_{R\uparrow}\psi_{R\uparrow}\psi^+_{L\downarrow}\psi_{L\downarrow}$

RHS $\Rightarrow \quad a\,\psi^+_{R\uparrow}\psi_{R\uparrow}\psi^+_{L\downarrow}\psi_{L\downarrow} + \frac{b}{4}\psi^+_{R\uparrow}\sigma^3_{\uparrow\uparrow}\psi_{R\uparrow}\psi^+_{L\downarrow}\sigma^3_{\downarrow\downarrow}\psi_{L\downarrow}$

$\Rightarrow -1/2 = a - \frac{b}{4} \quad \Rightarrow \quad a=-3/8,\ b=1/2$

for spin 1/2 system, we define $N^+ = \psi^+_{R\sigma}\psi_{L\sigma}$, $\vec{N}^+ = \psi^+_{R\alpha}\left(\frac{\vec{\sigma}}{2}\right)_{\alpha\beta}\psi_{L\beta}$

$$N^+ N = \psi^+_{R\sigma}\psi_{L\sigma}\psi^+_{L\sigma'}\psi_{R\sigma'} = -\psi^+_{R\sigma}\psi_{R\sigma'}\psi^+_{L\sigma'}\psi_{L\sigma} = -\{[\psi^+_{R\uparrow}\psi_{R\uparrow}+\psi^+_{R\downarrow}\psi_{R\downarrow}][\psi^+_{L\uparrow}\psi_{L\uparrow}+\psi^+_{L\downarrow}\psi_{L\downarrow}]/2$$

$$+[\psi^+_{R\uparrow}\psi_{R\uparrow}-\psi^+_{R\downarrow}\psi_{R\downarrow}][\psi^+_{L\uparrow}\psi_{L\uparrow}-\psi^+_{L\downarrow}\psi_{L\downarrow}] + \psi^+_{R\uparrow}\psi_{R\downarrow}\psi^+_{L\downarrow}\psi_{L\uparrow} + h.c\} = -\frac{J_R J_L}{2} - 2\vec{J}_R\cdot\vec{J}_L$$

$$\vec{N}^+\vec{N} = :\psi^+_{R\alpha}\psi_{L\beta}::\psi^+_{L\gamma}\psi_{R\delta}:\left(\frac{\sigma}{2}\right)^a_{\alpha\beta}\left(\frac{\sigma^a}{2}\right)_{\gamma\delta} = -\frac{1}{4}:\psi^+_{R\alpha}\psi_{R\delta}\psi^+_{L\gamma}\psi_{L\beta}:(2\delta_{\alpha\delta}\delta_{\beta\gamma}-\delta_{\alpha\beta}\delta_{\gamma\delta})$$

$$= -\frac{1}{2}J_R J_L + 1/4\,\psi^+_{R\alpha}\psi_{R\gamma}\psi^+_{L\gamma}\psi_{L\alpha} = -\frac{1}{2}J_R J_L + 1/4\left(\frac{J_R J_L}{2} + 2\vec{J}_R\cdot\vec{J}_L\right)$$

$$= -\frac{3}{8}J_R J_L + \frac{1}{2}\vec{J}_R\cdot\vec{J}_L$$

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how about for spin 3/2 system $N^+ = \psi^+_{R\alpha}\psi_{L\alpha}$, $N^a = \psi^+_{R\alpha}\left(\frac{\Gamma}{2}\right)_{\alpha\beta}\psi_{L\beta}$

$$N^+ N = \psi^+_{R\alpha}\psi_{L\alpha}\psi^+_{L\beta}\psi_{R\beta} = -\psi^+_{R\alpha}\psi_{R\beta}\psi^+_{L\beta}\psi_{L\alpha}$$

since its ~~sol~~ SU(4) singlet, it must be

$$N^+ N = a\,\psi^+_{R\alpha}\psi_{R\alpha}\psi^+_{L\beta}\psi_{L\beta} + b\left(\psi^+_{R\alpha}\frac{\Gamma^a_{\alpha\beta}}{2}\psi_{R\beta}\psi^+_{L\gamma}\frac{\Gamma^a_{\gamma\delta}}{2}\psi_{L\delta} + \psi^+_{R\alpha}\Gamma^{ab}_{\alpha\beta}\psi_{R\beta}\psi^+_{L\gamma}\Gamma^{ab}_{\gamma\delta}\psi_{L\delta}\right)$$

Chouse LHS $\alpha=\beta=3/2 \Rightarrow -\psi^+_{R3/2}\psi_{R3/2}\psi^+_{L3/2}\psi_{L3/2}$

$$RHS = a\,\psi^+_{R3/2}\psi_{R3/2}\psi^+_{L3/2}\psi_{L3/2} + \frac{b}{4}[\psi^+_{R3/2}\Gamma^4\psi_{R3/2}\psi^+_{L3/2}\Gamma^4\psi_{L3/2} + \psi^+_{R3/2}\Gamma^{15}\psi_{R3/2}\psi^+_{L3/2}\Gamma^{15}\psi_{L3/2}$$
$$+\psi^+_{R3/2}\Gamma^{23}\psi_{R3/2}\psi^+_{L3/2}\Gamma^{23}\psi_{L3/2}]$$

$$\Rightarrow -1 = a + \frac{b}{4}(1+1+1) \quad \text{i.e} \quad -1 = a + \frac{3b}{4}$$