Lect 6: Topological index for quantum Hall system

§ Magnetic translation for $H = \dfrac{(\vec{P} - \frac{e}{c}\vec{A})^2}{2m}$ with $\nabla \times \vec{A} = B\hat{z}$

1. symmetric gauge $\vec{A} = \frac{B}{2}\hat{z}\times\vec{r} = \frac{B}{2}(-y, x)$

$$T_x = e^{-i\delta_x \cdot (P_x + \frac{e}{c}A_x)/\hbar} = e^{-i\delta_x[-i\hbar\partial_x - \frac{eB}{2c}y]/\hbar}$$

$$T_y = e^{-i\delta_y (P_y + \frac{e}{c}A_y)/\hbar} = e^{-i\delta_y[-i\hbar\partial_y + \frac{eB}{2c}x]/\hbar}$$

2. Landau gauge $\vec{A} = -B(y, 0)$

$$T_x = e^{-i\delta_x P_x/\hbar} \qquad T_y = e^{-i\delta_y[P_y + \frac{eB}{c}x]/\hbar}$$

Please check that $[T_x, H] = [T_y, H] = 0$. for these two gauges.

Magnetic Bloch theorem:

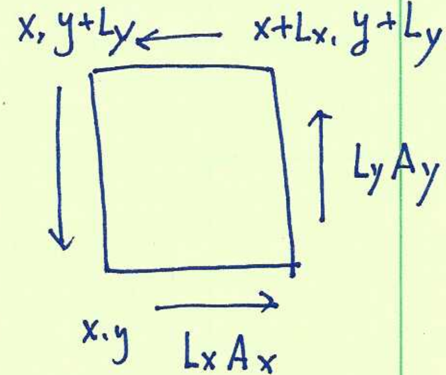
Define a magnetic unit cell with $L_x \times L_y$ which encloses a fundemental flux Φ_0. Then $\psi_{k_x,k_y}(x,y) = e^{ik_x x + ik_y y}\, u_{k_1k_2}(x,y)$

with
$$u_{k_1k_2}(x+L_x, y) = e^{i2\pi L_x A_x/\Phi_0}\, u_{k_1k_2}(x,y)$$
$$u_{k_1k_2}(x, L_y+y) = e^{i2\pi L_y A_y/\Phi_0}\, u_{k_1k_2}(x,y)$$

for the symmetric gauge.

HW: write down the magnetic Bloch theorem for the Landau gauge.

Define: $u_{k_1 k_2}(x,y) = |u_{k_1 k_2}(x,y)|\, e^{i\Theta_{k_1 k_2}(x,y)}$



winding number

$$P = \frac{+1}{2\pi}\oint d\vec{l}\; \nabla_l \Theta_{k_1,k_2}$$

— the number of zeros of u_{k_1,k_2}, i.e. vorticity.

$$\oint d\vec{l}\; \nabla_l \Theta_{k_1,k_2} = 2\pi\, \Phi_0/\Phi_0 = 2\pi \quad\Rightarrow\quad P = 1.$$

HW: ① Verify that P is gauge-independent.

② Consider solving the B-deG equation for superconducting vortex lattices. How many vortices are enclosed in a magnetic unit cell?

§ Kubo formula for Hall conductance

§ $$H = H_0 + \Delta H, \quad \text{where } \Delta H = -\int dv \frac{1}{c} \vec{j} \cdot \vec{A}$$

(here $\vec{A}$ is related to the external electric field $\vec{E}$ through $\vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t}$

$$\hat{\vec{j}}(\vec{r}) = -\frac{ie\hbar}{2m}\left(\psi^+(r)\nabla\psi(r) - \nabla\psi^+\psi\right).$$

In the interaction picture

$$\left.\begin{array}{l} |\psi(t)\rangle_I = T\exp\left[-\frac{i}{\hbar}\int_{t_0}^{t}\Delta H(t')dt'\right]|\psi(t_0)\rangle_I \approx \left(1 - \frac{i}{\hbar}\int_{t_0}^{t}\Delta H(t')dt'\right)|\psi(t_0)\rangle \\ \hat{\vec{j}}(\vec{r},t) = e^{iH_0t/\hbar}\,\hat{\vec{j}}(\vec{r})\,e^{-iH_0t} \end{array}\right\}$$

we set as $t_0 \to -\infty$, $|\psi(t_0)\rangle$ is the ground state of H_0

$$\Rightarrow \langle j(\vec{r},t)\rangle = \langle 0(t)|\,\hat{j}(\vec{r},t)\,|0(t)\rangle$$

$$= \langle 0|\left[1 + \frac{i}{\hbar}\int_{-\infty}^{t}\Delta H(t')dt'\right]\hat{j}(\vec{r},t)\left[1 - \frac{i}{\hbar}\int_{-\infty}^{t}\Delta H(t')dt'\right]|0\rangle$$

$$= \underbrace{\langle 0|\hat{j}(\vec{r},t)|0\rangle}_{=0} + \frac{i}{\hbar}\int_{-\infty}^{t}dt'\,\langle 0|[\Delta H(t'),\hat{j}(\vec{r},t)]|0\rangle$$

$$\Rightarrow \langle j(\vec{r},t)\rangle = -\frac{i}{\hbar}\int_{-\infty}^{t}dt'\,\langle 0|[\hat{j}(\vec{r},t),\Delta H(t')]|0\rangle$$

HW: ~~More precisely, the electric current formula above~~ doesn't include the diamagnetic part. The true current

$$\vec{j}(\vec{r}) = -\frac{ie\hbar}{2m}\left(\psi^\dagger(r)\nabla\psi(r) - \nabla\psi^\dagger\psi\right) - \frac{e^2}{mc}\psi^\dagger\psi\vec{A}$$

$$\Rightarrow \boxed{\vec{j}(\vec{r},t) = -\frac{i}{\hbar}\int_{-\infty}^{t} dt' \langle 0|[\hat{j}(\vec{r},t), \Delta H(t')|0\rangle - \frac{e^2}{mc} n\vec{A}}$$

$$\to \vec{j}(\vec{r},t) = \frac{i}{\hbar}\int d\vec{r}'\int_{-\infty}^{t} dt' \langle 0|[\hat{j}(\vec{r},t), \hat{j}(\vec{r}',t')]|0\rangle \cdot \frac{\vec{A}(\vec{r}',t')}{c} - \frac{ne^2}{mc}\vec{A}(\vec{r},t)$$

$$\to = \sum_{q'} \frac{i}{V\hbar c}\int_{-\infty}^{t} dt' \langle 0|[\hat{j}(\vec{r},t)\ \hat{j}_\alpha(-q',t')]|0\rangle A_\alpha(q',t') - \frac{ne^2}{mc}\vec{A}(\vec{r},t)$$

$$\Rightarrow j_\alpha(\vec{q},t) = \frac{i}{V\hbar c}\int_{-\infty}^{+\infty} dt' \underbrace{\theta(t-t')\langle 0|[\hat{j}_\alpha(q,t); j_\beta(-q,t')]|0\rangle}_{\text{only depend on } t-t'} A_\beta(q,t')$$

$$+ \frac{ne^2}{mc}A_\alpha(\vec{q},t)$$

Define $D_{\alpha\beta}(t-t') = \frac{i}{V\hbar}\theta(t-t')\langle 0|[\hat{j}_\alpha(q,t), j_\beta(-q,t')]|0\rangle$

then $$j_\alpha(\vec{q},t) = \frac{1}{c}\int_{-\infty}^{+\infty} dt' D_{\alpha\beta}(t-t') A_\beta(\vec{q},t') - \frac{ne^2}{mc}A_\alpha(\vec{q},t)$$

Define Fourier transform

$$j_\alpha(\vec{q},t) = \int\frac{d\omega}{2\pi} j_\alpha(\vec{q},\omega)e^{-i\omega t}$$

$$D_{\alpha\beta}(t-t') = \int \frac{d\omega}{2\pi} \; D_{\alpha\beta}(q,\omega)\, e^{-i\omega(t-t')}$$

$$A_\beta(q,t') = \int \frac{d\omega'}{2\pi} \; A_\beta(q,\omega')\, e^{-i\omega' t'}$$

$$\Rightarrow j_\alpha(\vec{q},t) = \frac{1}{c} \int_{-\infty}^{+\infty} dt' \int \frac{d\omega_1 d\omega_2}{(2\pi)^2} \; D_{\alpha\beta}(q,\omega)\, A_\beta(q,\omega')\, e^{-i\omega t} \; e^{i(\omega-\omega')t'}$$

$$- \frac{ne^2}{mc} A_\alpha(\vec{q},t)$$

$$= \frac{1}{c} \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \; D_{\alpha\beta}(q,\omega)\, A_\beta(q,\omega)\, e^{-i\omega t} \; - \frac{ne^2}{mc} \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \; A_\alpha(\vec{q},\omega)\, e^{-i\omega t}$$

$$\Rightarrow \; j_\alpha(\vec{q},\omega) = \left[D_{\alpha\beta}(q,\omega) - \frac{ne^2}{m} \delta_{\alpha\beta} \right] \frac{1}{c} A_\beta(\vec{q},\omega)$$

According to $\vec{E} = -\frac{1}{c} \partial_t \vec{A} \;\Rightarrow\; \vec{E}(\vec{q},\omega) = + \frac{i\omega}{c} \vec{A}(\vec{q},\omega)$

$$\Rightarrow j_\alpha(\vec{q},\omega) = \frac{1}{i\omega} \left[D_{\alpha\beta}(q,\omega) - \frac{ne^2}{m} \delta_{\alpha\beta} \right] E_\beta(\vec{q},\omega)$$

or $$\sigma_{\alpha\beta}(\vec{q},\omega) = \frac{1}{i\omega} D_{\alpha\beta}(q,\omega) + \frac{ne^2}{m\omega}\, i\, \delta_{\alpha\beta}$$

$$= \frac{1}{i\omega V} \int_{-\infty}^{+\infty} dt' \; \left(\frac{+i}{\hbar}\right) \theta(t-t') \langle 0 | [\hat{j}_\alpha(q,t), j_\beta(-q,t')] | 0 \rangle \, e^{i\omega(t-t')}$$

$$+ i \frac{ne^2}{m\omega} \delta_{\alpha\beta}$$

$$\boxed{\sigma_{\alpha\beta}(\vec{q},\omega) = \frac{1}{\hbar\omega V} \int_{-\infty}^{+\infty} dt' \; \theta(t-t')\, e^{i(\omega+i\eta)(t-t')} \langle 0 | [\hat{j}_\alpha(q,t), \hat{j}_\beta(-q,t')] | 0 \rangle + \frac{ine^2}{m\omega} \delta_{\alpha\beta}}$$

Now we use the Lehman representation

$$\sigma_{\alpha\beta}(\vec{q},\omega) = \frac{1}{\hbar\omega V}\int_0^{+\infty} dt \; e^{i\cancel{\hbar}(\omega+i\eta)t}\Big\{\langle 0|j_\alpha(q)|n\rangle\langle n|j_\beta(-q)|0\rangle e^{-i(E_n-E_0)t} - \langle 0|j_\beta(-q)|n\rangle\langle n|j_\alpha(q)|0\rangle e^{i(E_n-E_0)t}\Big\}$$

$$+ \frac{ine^2}{m\omega}\delta_{\alpha\beta}$$

$$= \frac{1}{\hbar\omega V}\sum_n \frac{-\langle 0|j_\alpha(q)|n\rangle\langle n|j_\beta(-q)|0\rangle}{i\left[\omega+i\eta-\frac{(E_n-E_0)}{\hbar}\right]} + \frac{\langle 0|j_\beta(-q)|n\rangle\langle n|j_\alpha(q)|0\rangle}{i\left[\omega+i\eta+\frac{E_n-E_0}{\hbar}\right]}$$

set $\vec{q}=0$,

$$\sigma_{\alpha\beta}(\omega) = \frac{i}{\omega V}\sum_n\left\{\frac{\langle 0|j_\alpha|n\rangle\langle n|j_\beta|0\rangle}{\hbar\omega+i\eta-(E_n-E_0)} - \frac{\langle 0|j_\beta|n\rangle\langle n|j_\alpha|0\rangle}{\hbar\omega+i\eta+(E_n-E_0)}\right\}$$

$$+\frac{ine^2}{m\omega}\delta_{\alpha\beta}$$

For Hall conductance, set $\omega\to 0$, $\alpha = x$ and $\beta = y$

$$\frac{1}{\hbar\omega-(E_n-E_0)} = \frac{-1}{E_n-E_0} - \frac{\hbar\omega}{(E_n-E_0)^2}$$

$$\frac{1}{\hbar\omega+(E_n-E_0)} = \frac{1}{E_n-E_0} - \frac{\hbar\omega}{(E_n-E_0)^2}$$

$$\Rightarrow \boxed{\sigma_{xy} = \frac{i\hbar}{V}\sum_{n\neq 0}\frac{\langle 0|j_y|n\rangle\langle n|j_x|0\rangle - \langle 0|j_x|n\rangle\langle n|j_y|0\rangle}{(E_n-E_0)^2}}$$

☆ Consider $H = \dfrac{(\vec{P} - \frac{e}{c}\vec{A})^2}{2m}$,

$\psi_{\alpha,\vec{k}}(\vec{r}) = e^{ik_1x + ik_2y}\, u_{\alpha,k}(\vec{r})$, with $H\,\psi_{\alpha,\vec{k}}(\vec{r}) = E_\alpha(\vec{k})\,\psi_{\alpha,\vec{k}}(\vec{r})$

$$\Rightarrow \underbrace{\left[e^{-i\vec{k}\cdot\vec{r}}\, H\, e^{i\vec{k}\cdot\vec{r}}\right]}_{\hat{H}(\vec{k})} u_{\alpha,k}(\vec{r}) = E_\alpha(\vec{k})\,\psi_{\alpha,\vec{k}}(\vec{r})$$

$$\Rightarrow \boxed{\hat{H}(\vec{k})\, u_{\alpha,\vec{k}}(\vec{r}) = E_\alpha(\vec{k})\, u_{\alpha,\vec{k}}(\vec{r})}$$

α - band index, $\vec{k} = (k_x, k_y)$ labels states in a magnetic Brillouin zone.

$$\hat{\psi}(r) = \sum_{k,\alpha} e^{i\vec{k}\cdot\vec{r}}\, u_{\alpha,\vec{k}}(\vec{r})\, C_\alpha(k)$$

field operator

$$\Rightarrow \hat{j}(\vec{r}) = \frac{e}{2m}\left[\hat{\psi}^\dagger(r)\left[-i\hbar\nabla - \frac{e}{c}A(r)\right]\psi(r)\right] - \left[-i\hbar\nabla + \frac{e}{c}A(r)\right]\psi^\dagger(r)\,\psi(r)]$$

$$\hat{j}(q=0) = \int d^3\vec{r}\; \hat{j}(\vec{r}) = \int d^3\vec{r} \sum_{k,\alpha}\sum_{k'\beta'} \frac{e}{m}\left[e^{-i\vec{k}\cdot\vec{r}}\, u^*_{\alpha,k}(r)\left(-i\hbar\nabla - \frac{e}{c}A\right)\right.$$

$$\left. e^{i\vec{k}'\cdot\vec{r}}\, u_{\beta,k'}(r)\right]\, C^\dagger_{\alpha,k}\, C_{\beta,k}$$

$$= e\sum_{k\alpha,k'\beta'} C^\dagger_{\alpha k}\, C_{\beta k}\, \langle k\alpha|\,\hat{v}\,|k'\beta\rangle$$

where $|k\alpha\rangle = e^{i\vec{k}\cdot\vec{r}}\, u_{\alpha,k}(\vec{r})$

and $\langle k\alpha|\,\hat{v}\,|k'\beta\rangle = \int d^3r\, \left(u_{k\alpha}(\vec{r})e^{i\vec{k}\cdot\vec{r}}\right)^*\left(\frac{-i\hbar\nabla}{m} - \frac{e}{mc}\vec{A}\right)\left(e^{i\vec{k}'\cdot\vec{r}}\, u_{\beta k'}(\vec{r})\right)$

Since magnetic translation $\hat{T}_R$ commutes with $\hat{v}$ - velocity operator,

$$[\hat{T}_R , \hat{v}] = 0 \quad \Rightarrow \quad \hat{T}_R \left(\hat{v} |\underset{\hat{\alpha}}{k}\rangle\right) = \hat{v}\, \hat{T}_R |\underset{\hat{\alpha}}{k}\rangle = e^{i\vec{k}\cdot\vec{R}} \left(\hat{v}|\underset{\hat{\alpha}}{k}\rangle\right)$$

$$\Rightarrow \quad \langle k\alpha | \hat{v} | k'\beta\rangle = \delta_{kk'} \langle k\alpha | \hat{v} | k\beta\rangle$$

where $\langle k\alpha | \hat{v} | k\beta\rangle = \int d^3r \; u^*_{k\alpha}(\vec{r})\, e^{-ikr} \left[-\frac{i\hbar\nabla}{m} - \frac{e}{mc}\vec{A}\right] \left(e^{ikr} u_{k\beta}(\vec{r})\right)$

$$= \int d^3r \; u^*_{k\alpha}(\vec{r}) \left[-\frac{i\hbar\nabla}{m} + \frac{\hbar\vec{k}}{m} - \frac{e}{mc}\vec{A}\right] u_{k\beta}(\vec{r})$$

$$= \int d^3r \; u^*_{k\alpha}(\vec{r}) \frac{\partial H(\vec{k})}{\partial \vec{k}} u_{k\beta}(\vec{r}) \longleftarrow \text{Exercise : prove it}$$

$$= \boxed{\langle u_{k\alpha} | \frac{\partial H(\vec{k})}{\partial \vec{k}} | u_{k\beta}\rangle = \langle k\alpha | \hat{v} | k\beta\rangle}$$

For non-interacting system : $|n\rangle = C^\dagger_{\alpha,k} C_{\beta,k} |0\rangle \leftarrow$ since $j(q=0)$ conserved $\vec{k}$

$$\Rightarrow \sigma_{xy} = \frac{i\hbar e^2}{V} \sum_{\substack{\cancel{\alpha\, E_\alpha(k) > E_F > E_\beta(k)} \\ E_\beta(k) > E_F > E_\alpha(k)}} \frac{\langle k\alpha | v_y | k\beta\rangle \langle k\beta | v_x | k\alpha\rangle - \langle k\alpha | v_x | k\beta\rangle \langle k\beta | v_y | k\alpha\rangle}{(E_\beta(k) - E_\alpha(k))^2}$$

$$= \frac{i\hbar e^2}{V\hbar^2} \sum_{E_\beta(k) > E_F > E_\alpha(k)} \frac{\left\{\langle u_{k\alpha} | \frac{\partial H(k)}{\partial k_y} | u_{k\beta}\rangle \langle u_{k\beta} | \frac{\partial H(k)}{\partial k_x} | u_{k\alpha}\rangle - \langle u_{k\alpha} | \frac{\partial H(k)}{\partial k_x} | u_{k\beta}\rangle \langle u_{k\beta} | \frac{\partial H(k)}{\partial k_y} | u_{k\alpha}\rangle\right\}}{(E_\beta(k) - E_\alpha(k))^2}$$

Exercise : $\langle u_{k\alpha} | \frac{\partial H(k)}{\partial k_x} | u_{k,\beta} \rangle = (E_k^\beta - E_k^\alpha) \langle u_{k,\alpha} | \frac{\partial}{\partial k} u_{k,\beta} \rangle$

Prove

$$= -(E_k^\beta - E_k^\alpha) \langle \frac{\partial u_{k\alpha}}{\partial k} | u^\beta \rangle$$

$$\Rightarrow \sigma_{xy} = \frac{i\hbar e^2}{V\hbar} \sum_{E_\beta(k) > E_F > E_\alpha(k)} \Big\{ \langle \frac{\partial u_{k\alpha}}{\partial k_y} | u_{k\beta} \rangle \langle u_{k\beta} | \frac{\partial u_{k\alpha}}{\partial k_x} \rangle$$

$$- \langle \frac{\partial u_{k\alpha}}{\partial k_x} | u_{k\beta} \rangle \langle u_{k\beta} | \frac{\partial u_{k\alpha}}{\partial k_y} \rangle \Big\}$$

using $\sum_{E^\alpha < E_F < E^\beta} |u_{k\alpha}\rangle\langle u_{k\alpha}| + |u_{k\beta}\rangle\langle u_{k\beta}| = 1 \leftarrow$ for a fixed k

$$\Rightarrow \sigma_{xy} = \frac{i\hbar e^2}{V\hbar} \sum_{k, \alpha < E_F < \beta} \langle \frac{\partial u_{k,\alpha}}{\partial k_y} | \frac{\partial u_{k\alpha}}{\partial k_x} \rangle - \langle \frac{\partial u_{k\alpha}}{\partial k_x} | \frac{\partial u_{k\alpha}}{\partial k_y} \rangle$$

$$\underbrace{- \langle \frac{\partial u_{k,\alpha}}{\partial k_y} | u_{k\alpha} \rangle \langle u_{k\alpha} | \frac{\partial u_{k\alpha}}{\partial k_x} \rangle + \langle \frac{\partial u_{k\alpha}}{\partial k_x} | u_{k\alpha} \rangle \langle u_{k\alpha} | \frac{\partial u_{k\alpha}}{\partial k_y} \rangle}$$

The 2nd line : $\langle \frac{\partial u_{k,\alpha}}{\partial k_y} | u_{k\alpha} \rangle = - \langle u_{k\alpha} | \frac{\partial u_{k\alpha}}{\partial k_y} \rangle \Rightarrow$ the 2nd line $= 0$

$$\langle u_{k\alpha} | \frac{\partial u_{k\alpha}}{\partial k_x} \rangle = - \langle \frac{\partial u_{k\alpha}}{\partial k_x} | u_{k,\alpha} \rangle$$

$$\Rightarrow \sigma_{x,y} = \frac{i\hbar e^2}{V\hbar} \sum_{\alpha < E_F} \sum_k \langle \frac{\partial u_{k\alpha}}{\partial k_y} | \frac{\partial u_{k\alpha}}{\partial k_x} \rangle - \langle \frac{\partial u_{k\alpha}}{\partial k_x} | \frac{\partial u_{k\alpha}}{\partial k_y} \rangle$$

$$= \frac{ie^2}{\hbar} \sum_{\alpha < E_F} \int \frac{d^2k}{(2\pi)^2} \left[\partial_{k_x} \langle u_{k\alpha} | \partial_{k_y} u_{k\alpha} \rangle - \partial_{k_y} \langle u_{k\alpha} | \partial_{k_x} | u_{k\beta} \rangle \right]$$

Define Berry connetion $\vec{A}_{\alpha}(k) = \langle u_{k,\alpha} | -i\vec{\partial}_k \, | u_{k,\beta} \rangle$

$$= \int d^2 r \;\; u^*_{k,\alpha} (-i\partial_k \, u_{k,\beta})$$

$\Rightarrow$

$$\sigma_{xy} = \frac{e^2}{h} \, \frac{1}{2\pi} \sum_{\alpha < E_F} \int \left[\partial_{k_x} A_{k_y}(\vec{k}) - \partial_{k_y} A_{k_x}(\vec{k}) \right] d^2 k$$

$$\boxed{\sigma_{xy} = \frac{e^2}{h} \sum_{\alpha < E_F} C_\alpha \, , \quad \text{with} \quad C_\alpha = \frac{1}{2\pi} \int d^2 k \left(\nabla_k \times \vec{A}(\vec{k}) \right)_{\hat{z}}}$$

* C_α: the Chern # ~~ass~~ associated with the α- band!

* Quantum mechanical WF can only be well-defined up to a phase. For a different phase convention

$$|u_{\vec{k}}\rangle \rightarrow |u'_{\vec{k}}\rangle = e^{i\theta(\vec{k})}\,|u_{\vec{k}}\rangle$$

It's easy to show that $\vec{A}'(\vec{k}) = \vec{A}(\vec{k}) + \partial_k \theta(\vec{k})$,

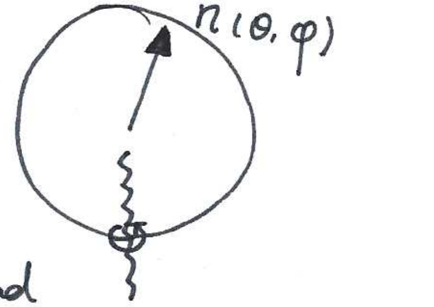
thus σ_{xy} is invariant.

* If $|u_{\vec{k}}\rangle$ is regularly defined over the entire BZ, then $\vec{A}(\vec{k})$ is also regular. Since BZ has no boundary, we have

$$C_\alpha = \frac{1}{2\pi}\int d^2k \left(\nabla_k \times \vec{A}(k)\right)_{\hat{z}} = 0.$$

* In order to have non-zero C_α, $|u_{\vec{k}}\rangle$ cannot be smoothly well-defined in the BZ. To understand it, let us use the spin-1/2 Berry phase example. The state $\hat{n}\cdot\vec{\sigma}\,|\psi(\hat{n})\rangle = |\psi(\hat{n})\rangle \rightarrow$

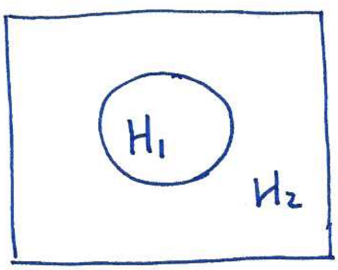
$$|\psi(\hat{n})\rangle = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}\,e^{i\varphi} \end{pmatrix}$$



we define $|\psi(\hat{n})\rangle$ in the following convention that $\langle\uparrow|\psi(\hat{n})\rangle > 0$. Nevertheless, there's a bad point at $\hat{n}$ on the south pole, where $\cos\frac{\theta}{2} = 0$. As a result, the phase of $|\psi(\hat{n})\rangle$ is not well-defined! Look at the $\downarrow$-component — "φ" is not defined

This corresponds to the gauge that a Dirac string located at the south pole. Around the Dirac string, phase varies 2π.

* For the case $|u_{\vec{k}}^{(\alpha)}\rangle$, we can choose a location $\vec{r}_0$ insiden the magnetic unit cell. The phase convention is that $\langle \vec{r}_0 | u_{\vec{k}}^{(\alpha)} \rangle > 0$. Nevertheless, if there exist a point $\vec{k}_0$, such that $\langle \vec{r}_0 | u_{\vec{k}_0}^{(\alpha)} \rangle = 0$, then the phase of $|u_{\vec{k}_0}^{(\alpha)}\rangle$ cannot be determined. Then $\sigma_{xy}^{(\alpha)}$ is given by the total vorticity of $\langle \vec{r}_0 | u_k^{(\alpha)} \rangle$ as a function of k lying the BZ. As an example, consider the case that $\langle \vec{r}_0 | u_k^{(\alpha)} \rangle$ only vanishes at one point $\vec{k}_0$ in the BZ. We divide the BZ into two parts S_1 and S_2



① inside H_1, $\langle \vec{r}_1 | u_k^{(\alpha)} \rangle > 0$, the phase of $|u_k^{(\alpha)}\rangle$ is well-defined with respect to a point $\vec{r}_1$

② inside H_2, $\langle \vec{r}_0 | u_k^{(\alpha)} \rangle > 0$, $|u_k^{(\alpha)}\rangle$'s phase well-define respect to $\vec{r}_0$.

This corresponds to different choices of gauge. Hower, there's no a gauge the $\vec{A}$ can be defined smoothly and uniquely.

At the boundary H_1 and H_2, there's a phase mismatch

$$|u_{\vec{k}}^{II}\rangle = e^{i\chi(\vec{k})}\,|u_{\vec{k}}^{I}\rangle,$$ where $\chi(\vec{k})$ is a smooth

function on ∂H.

On ∂H, $\vec{A}_{II}(\vec{k}) = \vec{A}_{I}(\vec{k}) + \nabla_k \chi(\vec{k})$

$$\Rightarrow \sigma_{xy}^{(\alpha)} = \frac{e^2}{h}\frac{1}{2\pi}\left[\int_{H_1} d^2k\,(\nabla_k \times \vec{A}_I)_3 + \int_{H_{II}} d^2k\,(\nabla_k \times \vec{A}_{II})_3\right]$$

$$= \frac{e^2}{h}\frac{1}{2\pi}\oint_{\partial H}\left[A_I(\vec{k}) - A_{II}(\vec{k})\right]dk = \frac{e^2}{h}\frac{1}{2\pi}\oint \nabla_k \chi \cdot d\vec{k}$$

$$= \frac{e^2}{h}n$$ where $n = \frac{1}{2\pi}\oint_{\partial H} d\vec{k}\cdot\nabla_k \chi$ ← the winding number