Lect 5: Landau level in the symmetric gauge

magnetic translation, ~~glide~~ guiding center, etc

§ Symmetric gauge: $\vec{B} \parallel \hat{z}$ $\quad \vec{A} = \frac{1}{2}\vec{r}\times\vec{B} = \frac{B}{2}(y, -x)$

$$H_{2D}^{LL} = \frac{\left(\vec{p} - \frac{e}{c}\vec{A}\right)^2}{2m}$$

cyclotron radius $l_B = \sqrt{\frac{\hbar c}{|eB|}} = \frac{257\text{Å}}{\sqrt{B/\text{Tesla}}}$

$$= \frac{p^2}{2m} + \frac{1}{2}m\omega_0^2 r^2 \mp \frac{eB}{2mc}\hat{z}\cdot(\vec{r}\times\vec{p}) \qquad \omega_0 = \frac{|eB|}{2mc}$$

$$\boxed{H_{2D}^{LL} = \frac{p^2}{2m} + \frac{1}{2}m\omega_0^2 r^2 \mp \omega_0 L_z,}$$

where "$\mp$" apply for $eB < 0$, > 0, respectively.

Recap of the spectra and wavefunction of 2D harmonic oscillator

$$H_{har} = \frac{p^2}{2m} + \frac{1}{2}m\omega_0^2 r^2$$

$$\psi(r, \varphi) = R(r)\, e^{im\varphi}$$ magnetic quantum

$$E_{n_r, m} = (2n_r + |m| + 1)\hbar\omega_0$$

$$\psi_{n_r, m} = e^{im\varphi}\, r^{|m|}\, e^{-\frac{r^2}{2l_B^2}}\, F(-n_r, |m|+1, r^2/l^2)$$

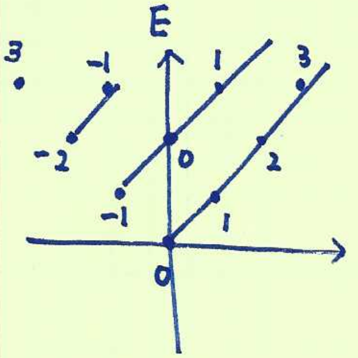
$$F(\alpha, \gamma, z) = 1 + \frac{\alpha}{\gamma}z + \frac{\alpha(\alpha+1)}{2!\,\gamma(\gamma+1)}z^2 + \cdots \frac{\alpha(\alpha+1)\cdots(\alpha+n-1)}{n!\,\gamma(\gamma+1)\cdots(\gamma+n-1)}z^n + \cdots$$

$$F(-n_r, |m+1|, (r/l)^2) = 1 + \frac{-n_r}{|m|+1}\left(\frac{r}{l}\right)^2 + \cdots \frac{(-n_r)(-n_r+1)\cdots(-)}{n_r!\,(|m+1|)\cdots(|m|+n_r)}\left(r/l\right)^{n_r}$$

For $H^{LL}_{2D} = \frac{p^2}{2m} + \frac{1}{2} m\omega_0^2 r^2 - \omega_0 L_z$

$$E_{Landau} = (2n_r + |m'| + 1 - m')\hbar\omega_0 = \begin{cases} \hbar\omega_c [n_r + 1/2] & \text{if } m > 0 \\ \hbar\omega_c [n_r + |m| + 1/2] & \text{if } m < 0 \end{cases}$$

with $\omega_c = 2\omega_0 = \frac{|eB|}{mc}$.



LLL – lowest Landau level

$$\boxed{\psi^{LLL}_{n_r=0,m} = \frac{1}{\sqrt{2\pi l_B^2 2^m m!}} \left(\frac{z}{l_B}\right)^m e^{-\frac{|z|^2}{4l_B^2}}}$$

$z = x + iy$, infinite degeneracy $m = 0, 1, 2, \cdots +\infty$

* classic orbit radius

$$\rho = |\psi|^2 \propto r^{2m} e^{-\frac{r^2}{2l_B^2}} \Rightarrow \frac{\partial\rho}{\partial r^2} = 0 \Rightarrow r_c^2 = 2m l_B^2$$

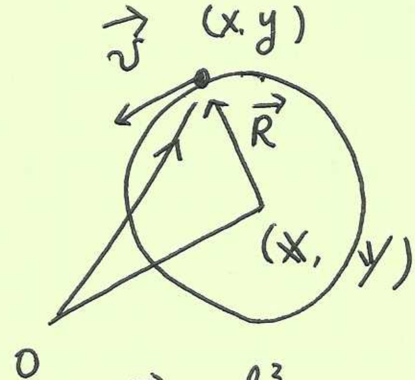
average density $\rho \sim \frac{1}{\pi(r_c^2(m+1) - r_c^2(m))} = \frac{1}{2\pi l_B^2}$

More exactly,

$$\rho(r) = \sum_{m=0}^{\infty} |\psi_{LLL,m}(r)|^2 = \frac{1}{2\pi l_B^2}\left[\sum_{m=0}^{\infty} \frac{1}{m!}\left(\frac{|z|^2}{2l_B}\right)^m\right] e^{-\frac{|z|^2}{2l_B}}$$

$$= \frac{1}{2\pi l_B^2}$$

* Magnetic translation



for $eB<0$

mechanical momenta $\vec{\mathbb{P}} = \vec{p} - \frac{e}{c}\vec{A}$

$$\mathbb{P}_x = -i\hbar\partial_x - \frac{eB}{2c} y = -i\hbar\partial_x + \frac{\hbar y}{2\ell_B^2}$$

$$\mathbb{P}_y = -i\hbar\partial_y + \frac{eB}{2c} x = -i\hbar\partial_y - \frac{\hbar x}{2\ell_B^2}$$

$$\vec{R} = \frac{\ell_B^2}{\hbar}(\mathbb{P}_y, -\mathbb{P}_x)$$

with $\frac{\ell_B^2}{\hbar} = \frac{c}{|eB|} = -\frac{c}{eB}$

Define guilding center

$$\mathbb{X} = x + \frac{\ell_B^2}{\hbar}\mathbb{P}_y = \frac{\ell_B^2}{\hbar}\left(-i\hbar\partial_y + \frac{\hbar}{2\ell_B^2} x\right)$$

$$\mathbb{Y} = y - \frac{\ell_B^2}{\hbar}\mathbb{P}_x = \frac{\ell_B^2}{\hbar}\left(-i\hbar\partial_x - \frac{\hbar}{2\ell_B^2} y\right)$$

$$\boxed{\text{HW: Check that guiding centers are conserved!}}$$

Proof: $[\mathbb{X}, \mathbb{P}_x] = \frac{\ell_B^2}{\hbar}\left\{\left[-i\hbar\partial_y, \frac{\hbar y}{2\ell_B^2}\right] + \frac{\hbar}{2\ell_B^2}[x, -i\hbar\partial_x]\right\}$

$$= \frac{-i}{2}\left\{[\partial_y, y] + [x, \partial_x]\right\} = 0$$

$[\mathbb{Y}, \mathbb{P}_y] = 0$ similarly

$[\mathbb{X}, \mathbb{P}_y] = [\mathbb{Y}, \mathbb{P}_x] = 0$

Since $H = \frac{\mathbb{P}_x^2 + \mathbb{P}_y^2}{2m} \Rightarrow [\mathbb{X}, H] = [\mathbb{Y}, H] = 0.$

Then we put guilding center onto expeniential

$$T_x(\delta_x) = e^{-iy\delta_x/\ell_B^2} = e^{-i(-i\partial_x \mp \frac{y}{2\ell_B^2})\cdot\delta_x} = e^{-\delta_x\partial_x \pm i\frac{y\delta_x}{2\ell_B^2}}$$

$$T_y(\delta_y) = e^{+ix\delta_y/\ell_B^2} = e^{i(i\partial_y \mp \frac{1}{2\ell_B^2}x)\delta_y} = e^{-\delta_y\partial_y \mp i\frac{x\delta_y}{2\ell_B^2}}$$

when applied to a wavefunction

$$T_x(\delta_x)\,\psi(x,y) = e^{-\delta_x\partial_x + i\frac{y\delta_x}{2\ell_B^2}}\,\psi(x,y) = e^{+i\frac{y\delta_x}{2\ell_B^2}}\,\psi(x-\delta_x,y)$$

$$T_y(\delta_y)\,\psi(x,y) = e^{-\delta_y\partial_y \mp i\frac{x\delta_y}{2\ell_B^2}}\,\psi(x,y) = e^{-i\frac{x\delta_y}{2\ell_B^2}}\,\psi(x,y-\delta_y)$$

More generally $T[\vec{\delta}] = e^{-\vec{\delta}\cdot\vec{\nabla} + \frac{i}{2\ell_B^2}\hat{z}\cdot(\vec{\delta}\times\vec{r})}$

and $\left[T[\vec{\delta}],\ H^{LL}_{2D}\right] = 0$

* Magnetic translation doesn't commute

$$T_x[\delta_x]\,T_y[\delta_y] = e^{\delta_x[-\partial_x \pm \frac{iy}{2\ell_B^2}]}\,e^{\delta_y[-\partial_y \mp \frac{ix}{2\ell_B^2}]}$$

according to $e^A e^B = e^B e^A e^{[A,B]}$ if $[[A,B],A]=[[A,B],B]=0$

$$\left[-\partial_x + \frac{iy}{2\ell_B^2},\ -\partial_y \mp \frac{ix}{2\ell_B^2}\right] = +\frac{i}{\ell_B^2}$$

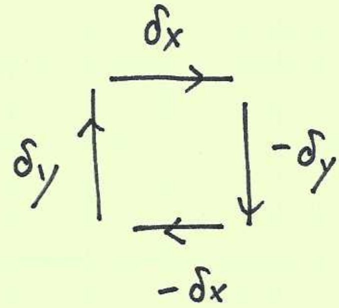
$$\Rightarrow\quad T_x[\delta_x]\,T_y[\delta_y] = T[\delta_y]\,T[\delta_x]\,e^{+i\frac{\delta_x\delta_y}{\ell_B^2}}$$

$$T_x^{-1}[\delta_x]\, T_y^{-1}[\delta_y]\, T_x[\delta_x]\, T_y[\delta_y] = e^{i\frac{\delta_x\delta_y}{\ell_B^2}}$$

$$= e^{i\frac{\delta_x\delta_y}{\frac{hc}{|eB|}}2\pi} = e^{-i2\pi\frac{B\delta_x\delta_y}{\frac{hc}{e}}} = e^{-i2\pi\,\Phi/\Phi_0}$$

$$\phi_0 = \frac{hc}{e}$$

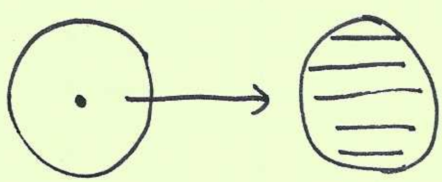
$$eB < 0$$



perform the translation to the Gaussian

$$T_x[\delta]\, e^{-\frac{|z|^2}{4\ell_B^2}} = e^{-\delta\partial_x + \frac{i}{2\ell_B^2}\delta y}\, e^{-\frac{|z|^2}{4\ell_B^2}}$$

$$= e^{-\frac{(x-\delta)^2+y^2}{4\ell_B^2}}\, e^{\frac{i}{2\ell_B^2}\delta y} = e^{\frac{-|z|^2}{4\ell_B^2}}\, \underbrace{e^{\frac{\delta}{2\ell_B^2}(x+iy)}}_{f(z)\ \text{analytic function}}$$



$$\cdot\ e^{i\frac{y\delta}{2\ell_B^2}}$$

QHE

$$H = \frac{1}{2m}\left(-i\hbar\vec{\nabla} + \frac{e}{c}\vec{A}\right)^2 , \qquad \Pi_x = -i\hbar\partial_x + \frac{e}{c}A_x$$

$$[\Pi_x, \Pi_y] = -i\hbar\frac{e}{c}[\partial_x A_y - \partial_y A_x] = -\frac{i\hbar e B}{c} = -i\hbar^2/\ell^2 , \quad \ell^2 = \frac{\hbar c}{eB}$$

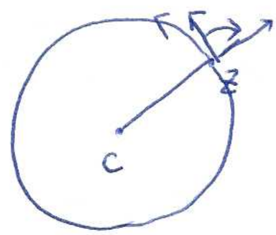
$$\Rightarrow \quad 2\pi\ell^2 B = \phi_0 :$$

define operator $a^\dagger = \frac{\ell/\hbar}{\sqrt{2}}(\Pi_x + i\Pi_y)$, $a = \frac{\ell/\hbar}{\sqrt{2}}(\Pi_x - i\Pi_y)$

$$[a, a^\dagger] = \frac{1}{2}(\ell/\hbar)^2(-i)[\Pi_x, \Pi_y]\cancel{\times 2} = (\ell/\hbar)^2\left(\frac{\hbar}{\ell}\right)^2 = 1$$

$$\Rightarrow H = \frac{\hbar\omega_c}{2}(aa^\dagger + a^\dagger a)$$

The center of the cyclotron orbital



$$z - C = \frac{-i\Pi}{m\omega_c}$$

$$\Rightarrow C_x = x - \Pi_y/m\omega_c \qquad C_y = y + \Pi_x/m\omega_c$$

$$[C_x, C_y] = \frac{1}{2m\omega_c}[x, \Pi_x] + \frac{1}{2m\omega_c}[y, \Pi_y] = \frac{i\hbar}{m\omega_c}\times 2 - (i\hbar^2/\ell^2)\left(\frac{1}{m\omega_c}\right)^2$$

$$- [\Pi_y, \Pi_x]/(m\omega_c)^2$$

$\Rightarrow$

$$\omega_c = \frac{eB}{mc} \Rightarrow \frac{\hbar}{m\omega_c} = \frac{\hbar c}{eB} = \ell^2 \Rightarrow [C_x, C_y] = i(\ell^2\times 2 - \ell^2) = i\ell^2$$

$$b = \frac{1}{\sqrt{2}\ell}(C_x - iC_y) \Rightarrow [b, b^\dagger] = 1$$

defin $b^\dagger = \frac{1}{\sqrt{2}\ell}(C_x + iC_y)$

$$\Rightarrow \left.\begin{array}{l} [\Pi_x, C_x] = [\Pi_x, x] - \frac{1}{m\omega_c}[\Pi_y, \Pi_y] = -i\hbar + i\hbar^2/\ell^2 \, \frac{1}{m\omega_c} = 0 \\ [\Pi_x, C_y] = [\Pi_x, y] = 0 \quad \text{and similary } [\Pi_y, C_y] = [\Pi_y, C_x] = 0 \end{array}\right\} \Rightarrow [a_{a^\dagger}, b_{b^\dagger}] = 0$$

T

for the symmetric gauge, $\vec{A} = \frac{B}{2}(-y, x, 0)$

$$\Rightarrow \quad b = \frac{1}{\sqrt{2}}\left(\frac{z}{2l} + 2l\frac{\partial}{\partial \bar{z}}\right) \qquad a^{\dagger} = \frac{i}{\sqrt{2}}\left(\frac{z}{2l} - 2l\frac{\partial}{\partial \bar{z}}\right)$$

$$b^{\dagger} = \frac{1}{\sqrt{2}}\left(\frac{\bar{z}}{2l} - 2l\frac{\partial}{\partial z}\right) \qquad a = \frac{-i}{\sqrt{2}}\left(\frac{\bar{z}}{2l} + 2l\frac{\partial}{\partial z}\right)$$

The set of eigenstates

$$|n,m\rangle = \frac{(a^{\dagger})^n (b^{\dagger})^m}{\sqrt{n!\,m!}} |00\rangle$$

$$\psi_{00} \Rightarrow \left.\begin{aligned} &\left(\frac{z}{2l} + 2l\frac{\partial}{\partial \bar{z}}\right)\psi_{00} = 0 \\ -i&\left(\frac{\bar{z}}{2l} + 2l\frac{\partial}{\partial z}\right)\psi_{00} = 0 \end{aligned}\right\}$$

$$\Rightarrow \psi_{00} = \frac{1}{\sqrt{2\pi l^2}} e^{-z\bar{z}/4l^2}$$ satisfying the normalization condition

$$\psi_{0m} = \frac{(b^{\dagger})^m}{\sqrt{m!}}\left(\frac{1}{\sqrt{2\pi l^2}} e^{-\frac{z\bar{z}}{4l^2}}\right) = \frac{1}{\sqrt{m!}}\frac{1}{\sqrt{2\pi l^2}}\frac{1}{\sqrt{2^m}}\left(\frac{\bar{z}}{l}\right)^m e^{-\frac{z\bar{z}}{4l^2}}$$

lowest Landau level

$$z = \sqrt{2}\ell(b + a^\dagger), \qquad \bar{z} = \sqrt{2}\ell(b^\dagger + a)$$

$$\Rightarrow \langle nm|\bar{z}z|nm\rangle = 2\ell^2 \langle nm| b^\dagger b + a^\dagger a + 1|nm\rangle = 2\ell^2(1+n+m)$$

for large m. $|nm\rangle$ is localized to a ring with $r_{m,n} = \ell\sqrt{2(1+m+n)}$

and width of ℓ.

If the LLL is filled. particle density

$$\sum_{m=0}^{\infty} |\psi_{0,m}(z)|^2 = (2\pi\ell^2)^{-1} \sum_{m=0}^{N-1} \frac{x^m}{m!} e^{-x} = \frac{1}{2\pi\ell^2} \quad \text{uniform density}$$