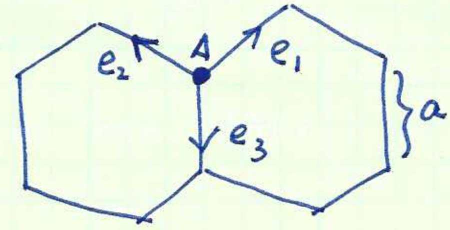
Lect 10 honeycomb lattice — graphen, haldane

graphene

$$H = -t \sum_{i\in A} \left(C^{\dagger}_{i+\hat{e}_j} C_i + h.c. \right)$$



$$\rightarrow \quad H = \sum \left(C^{\dagger}_A(k)\ C^{\dagger}_B(k) \right) H(k) \begin{pmatrix} C_A(k) \\ C_B(k) \end{pmatrix}$$

1. using the Fourier transform

$$\begin{cases} C_{i\in A} = \dfrac{1}{\sqrt{N}} \sum_k e^{i\vec{k}\cdot\vec{R}_i} C_A(k) \\ C_{j\in B} = \dfrac{1}{\sqrt{N}} \sum_k e^{i\vec{k}\cdot\vec{R}_j} C_B(k) \end{cases}$$

$$\Rightarrow H(k) = \vec{h}(k)\cdot\vec{\tau}$$

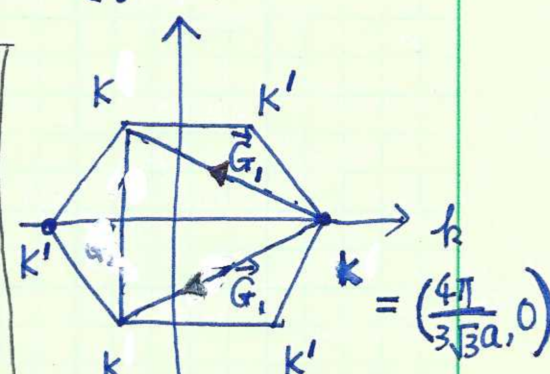
$\vec{h}(k)$ is planar :

$$\begin{cases} h_x(k) = \sum_{j=1}^{3} \cos \vec{k}\cdot\hat{e}_j \\ h_y(k) = \sum_{j=1}^{3} \sin \vec{k}\cdot\hat{e}_j \end{cases}$$

$h(k)$ is not periodic under shift of Reciprocal lattic vectors $\vec{G}_1$ and $\vec{G}_2$: $k \rightarrow \vec{k}+\vec{G}_1,\ \vec{k}+\vec{G}_2$.

HW: work out the relation between

$$H(k+\vec{G}_i) = U_i^{\dagger} H(k)\, U_i$$



② if we use
$$\begin{cases} C_{i\in A} = \frac{1}{\sqrt{N}} \sum_k e^{i\vec{k}\cdot\vec{R}_i} C_A(k) \\ C_{j\in B} = \frac{1}{\sqrt{N}} \sum_k e^{i\vec{k}\cdot\vec{R}_i} C_B(k) \end{cases}$$

then
$$H(k) = \begin{bmatrix} 0, & 1 + e^{i\vec{k}\cdot(\vec{e}_2-\vec{e}_3)} + e^{i\vec{k}\cdot(\vec{e}_1-\vec{e}_3)} \\ c.c., & 0 \end{bmatrix}$$

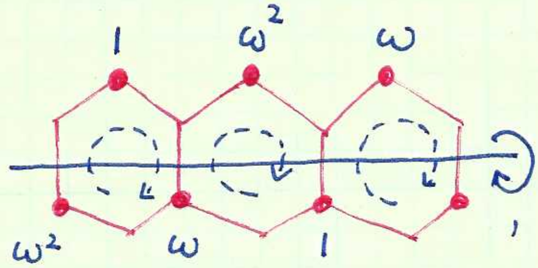
Then $H(k)$ is periodic

HW: Vanishing of $h(k)$ at K and K' — Prove this.

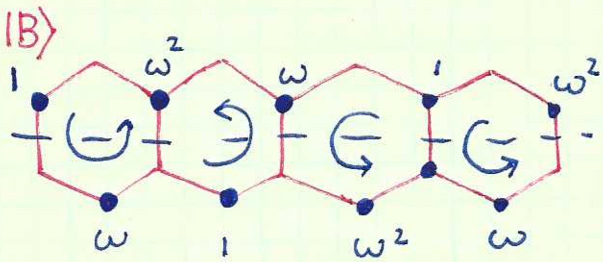
Check the little group at K and K' is D_3, which is non-abelian

Check the 2-fold degnerate state at K.

$|A\rangle$



$|B\rangle$



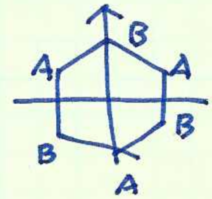
$\omega = e^{i\frac{2}{3}\pi}$

HW: expand $H(k)$ around K and K'. Check the winding numbers around K and K' are opposite.

HW: If the binding strength along three bonds t_1, t_2, t_3 are not equal, find the condition that Dirac cones exist

Symmetry constraint to the $\vec{h}$ field:

① Time reversal (anti-unitary, not spin)



$\vec{k} \to -\vec{k}$, $\quad \tau_{1,3} \to \tau_{1,3}$, $\quad \tau_2 \to -\tau_2$

② Rotation $180°$, $A\,B \to B, A$

$\vec{k} \to -\vec{k}$, $\quad \tau_1 \to \tau_1$, $\quad \tau_{2,3} \to -\tau_{2,3}$

- Both TR and Rotation $180°$ $\Rightarrow$ τ_3 term must vanish, or $h_3 = 0$. $\Rightarrow$ $\vec{h}$ is a planar field.

- x-reflection $\quad k_y \to -k_y$, $\tau_1 \to \tau_1$, $\tau_{2,3} \to -\tau_{2,3}$

$\Rightarrow h_x(k_x, k_y) = h_x(k_x, -k_y)$, $\quad h_{y,z}(k_x, k_y) = -h_{y,z}(k_x, -k_y)$,

y-reflection $\quad k_x \to -k_x$, $\quad \tau_{1,2,3} \to \tau_{1,2,3}$ $\quad \Rightarrow \vec{h}(k_x, k_y) = \vec{h}(-k_x, k_y)$.

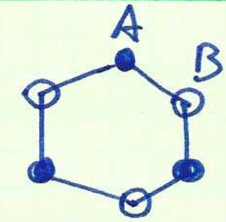
check: for $h_x = 1 + \cos \vec{k}\cdot(\vec{e}_2 - \vec{e}_3) + \cos \vec{k}\cdot(\vec{e}_1 - \vec{e}_3)$

$$= 1 + \cos\left[-\frac{\sqrt{3}k_x}{2} + \frac{3}{2}k_y\right] + \cos\left(\frac{\sqrt{3}k_x}{2} + \frac{3}{2}k_y\right) = 1 + 2\cos \sqrt{3}\, k_x \cos 3 k_y$$

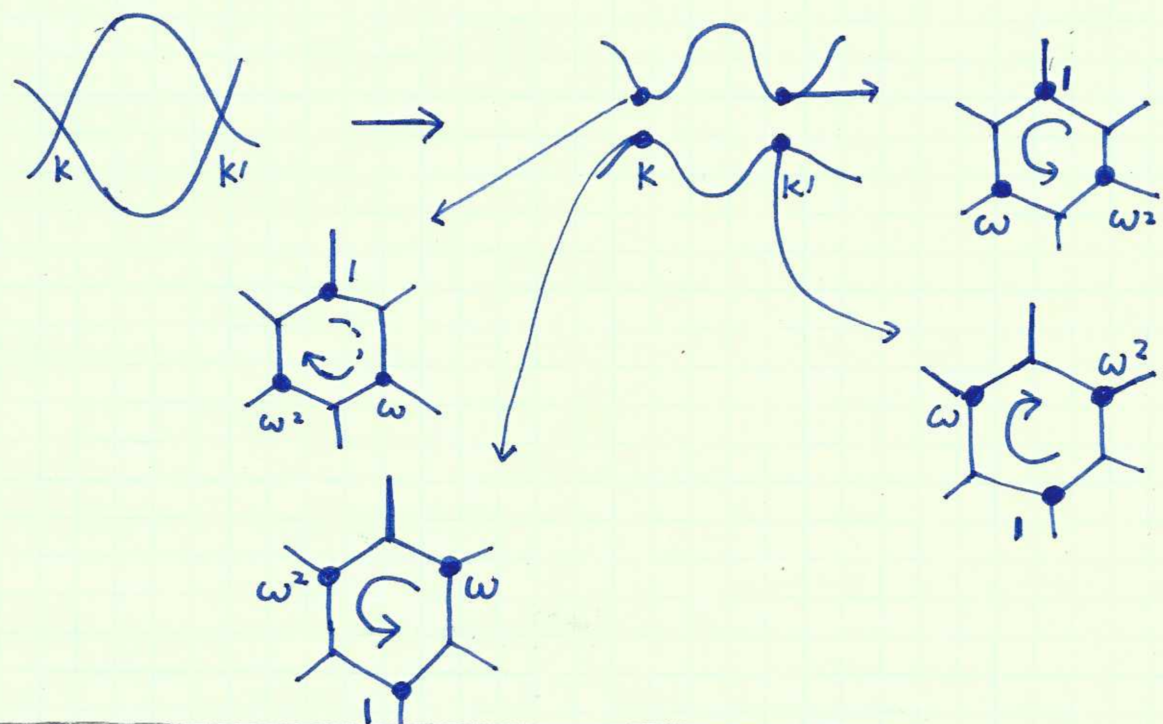
$$h_y = \sin \vec{k}\cdot(\vec{e}_2 - \vec{e}_3) + \sin \vec{k}\cdot(\vec{e}_1 - \vec{e}_3)$$

$$= \sin\left(-\frac{\sqrt{3}k_x}{2} + \frac{3}{2}k_y\right) + \sin\left(\frac{\sqrt{3}k_x}{2} + \frac{3}{2}k_y\right) = 2\sin 3 k_y \cos \sqrt{3}\, k_x .$$

gap opening — sub lattice asymmetry



$$H_M = m \sum_{\vec{r}} n_A(\vec{r}) - n_B(\vec{r}) \rightarrow h_z(\vec{r}) = m$$



Around $\vec{k} = \vec{K} + \vec{q} = \left(\frac{4\pi}{3\sqrt{3}a} + q_x, q_y\right) \Rightarrow h_x = 3q_x, \quad h_y = -3q_y$

$\vec{k} = \vec{K}' + \vec{q} = \left(-\frac{4\pi}{3\sqrt{3}a} + q_x, q_y\right) \Rightarrow h_x = -3q_x, \quad h_y = -3q_y$

Valley Hall effect

$$x = i\partial_k \longrightarrow \boxed{x = i\partial_{\vec{k}} + \vec{A}(\vec{k})}$$

HW: prove this

project to a band

$$H = \mathcal{E}_n(\vec{k}) + V(i\partial_k + \vec{A})$$

$$[x_i, x_j] = i[\partial_{k_i} A_j] + i[A_i, \partial_{k_j}] = i\epsilon_{ijk}\Omega_k$$

$\Rightarrow \hbar v_i = -i[x_i, H] = \nabla_k \mathcal{E}_n(\vec{k}) + (-i)[x_i, V(x)]$

$[x_i, V(x)] = [x_i, \quad x_j] \frac{\partial V}{\partial x_j} = \frac{\partial V}{\partial x_j} i \epsilon_{ijk} \Omega_k = i \epsilon_{ijk} \frac{\partial V}{\partial x_j} \Omega_k$

~~please verify~~ linearized potential

$\Rightarrow \hbar \vec{v} = \nabla_k \mathcal{E}_n(\vec{k}) + \frac{\partial V}{\partial \vec{x}} \times \vec{\Omega}_k$

$\rightarrow$ semi-classic equation

$$\begin{cases} \hbar \dot{x} = \nabla_k \mathcal{E} \boxed{- \dot{\vec{k}} \times \vec{\Omega}_k} \\ \hbar \dot{\vec{k}} = -\nabla V \boxed{+ e \dot{x} \times \vec{B}(r)} \end{cases}$$

Anomalous Hall current

Hall current.

Berry connection

$\vec{A}(k) = \langle \psi_k | i \partial_k | \psi_k \rangle \qquad \Omega_z(k) = \partial_{k_x} A_{k_y} - \partial_{k_y} A_{k_x}$

HW: prove if the system has TR symmetry, then

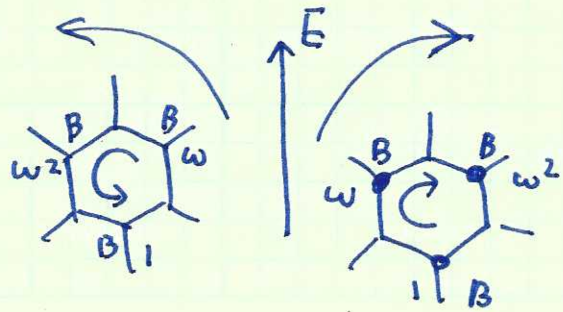
$\Omega_z(\vec{k}) = -\Omega_z(-\vec{k})$

$\vec{A}(k) = +\vec{A}(-k)$

The Berry curvature mainly distribute around $\vec{K}$ and $\vec{K}'$,

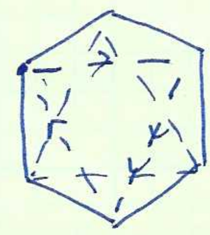
with $\Omega_z(\vec{K}) = -\Omega_z(-\vec{K}')$

Different valley are deflected oppositely



$\Rightarrow$ Valley Hall effect !!!

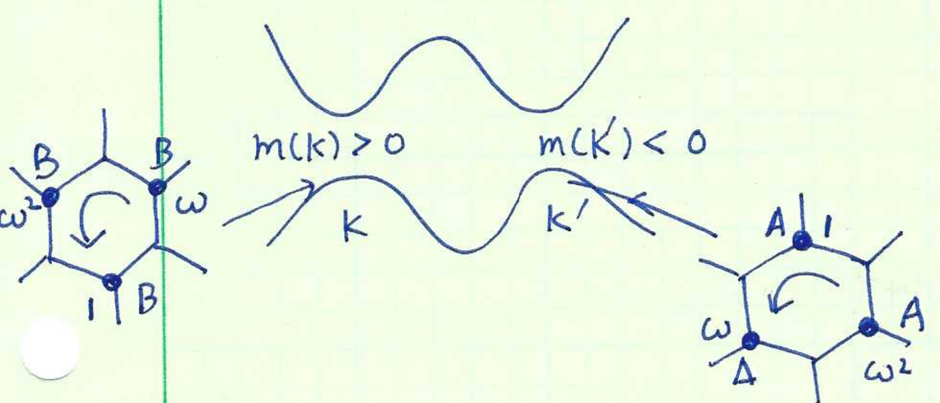
TR breaking terms



$$H^{o}_{haldane} = -t' \sum_{\langle\langle ij \rangle\rangle} C_i^{\dagger} C_j \, e^{i\delta} + h.c$$

$$\Rightarrow h_z(\vec{k}) = m(\vec{k}) = t' \sin\delta \sum_{ij} \sin \vec{k}\cdot(\hat{e}_i - \hat{e}_j)$$

$\Rightarrow$ the mass at $\vec{K}$ and $\vec{K}'$ are opposite



① TR symmetry is broken

② but Rotation 180° symmetry is preserved

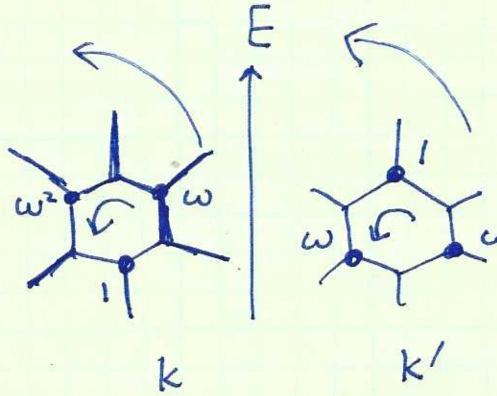
$$\boxed{\text{HW: prove that} \quad \vec{A}(\vec{k}) = -\vec{A}(-\vec{k}), \quad \Omega_z(\vec{k}) = \Omega_z(-\vec{k})}$$

~~For insulator~~ :

$$\sigma_{xy} = \frac{e^2}{h} \int \frac{dk_x dk_z}{2\pi} \Omega_z(\vec{k})\, n_f(\vec{k})$$

$\rightarrow$ insulator

$$\sigma_{xy} = \frac{e^2}{h} \int \frac{dk_x dk_y}{2\pi} \Omega_z(\vec{k}) = \frac{e^2}{h} C$$



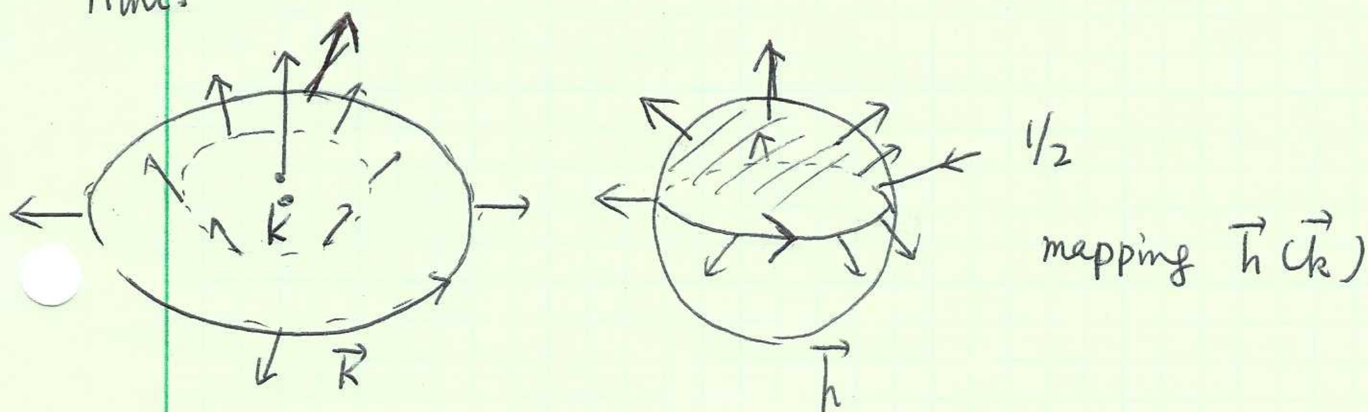
HW: prove that around the Dirac cone K, K'.

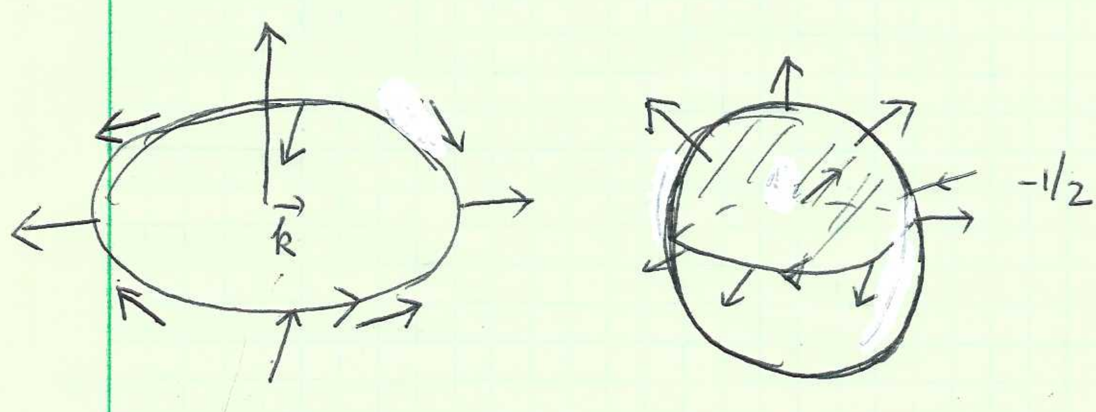
the Berry curvature $\rightarrow \displaystyle\iint_{\substack{\text{around}\\ K \text{ or } K'}} \frac{dk_x dk_z}{2\pi}\,\Omega_z \rightarrow \pm 1/2$.

For A. B sublattice asymmetry case $C \sim 1/2 - 1/2 = 0$

Haldane ~~twisted~~ mass case $C = 1/2 + 1/2 = 1$.

Hint:





Hall conductance for 2-band system

$$H(\vec{k}) = \varepsilon(\vec{k}) + \sigma_\alpha \cdot d_\alpha(\vec{k})$$

Matsubara Green's function $G(k,\tau) = -\langle T_\tau [\psi(k,\tau)\psi^\dagger(k,0)]\rangle$

$$\left\{ G(k,\tau) = \frac{1}{\beta}\sum_{i\omega_n} G(k,i\omega_n)\, e^{-i\omega_n \tau} \right.$$

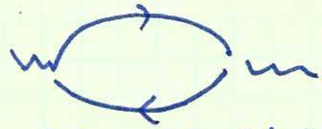
$$G(k,ik_n) = \frac{1}{2}\left(\frac{1+\vec{\sigma}\cdot\hat{d}(\vec{k})}{ik_n - (\varepsilon(\vec{k}) + d)} + \frac{1-\vec{\sigma}\cdot\hat{d}(\vec{k})}{ik_n - (\varepsilon(\vec{k}) - d)} \right)$$

$\hat{d}$ is the unit vector

Current operators

$$j_i(k) = \frac{\partial H(k)}{\partial k_i} = \frac{\partial \varepsilon}{\partial k_i} + \frac{\partial d_\alpha(k)}{\partial k_i}\sigma^\alpha$$

current - current correlation



$$Q_{ij}(q,i\omega_n) = \frac{1}{V\beta}\, tr\left[G(k+q/2, ik_n + i\omega_n)\, j_i\, G(k-q/2, ik_n)\, j_j \right]$$

$$= \frac{1}{4V\beta}\sum_{k,ik_n} tr \Bigg[\left(\frac{1+\vec{\sigma}\cdot\hat{d}(k+q/2)}{i\omega_n + ik_n - E_+(k+q/2)} + \frac{1-\vec{\sigma}\cdot\hat{d}(k+q/2)}{i\omega_n + ik_n - E_-(k+q/2)} \right)$$

$$\left(\frac{\partial \varepsilon(k)}{\partial k_i} + \frac{\partial d_\alpha}{\partial k_i}\sigma^\alpha \right) \left(\frac{1+\vec{\sigma}\hat{d}(k-q/2)}{ik_n - E_+(k-q/2)} + \frac{1-\vec{\sigma}\cdot\hat{d}(k-q/2)}{i\omega_n - E_-(k-q/2)} \right)$$

$$\left(\frac{\partial \varepsilon(k)}{\partial k_j} + \frac{\partial d_\beta}{\partial k_i}\sigma^\beta \right) \Bigg]$$

Set $q=0$, we divide the contribution into intra band and inter band transitions!

The intra band contributions are zero for insulators

- Check the frequency summation

$$\sum_{ik_n} \frac{1}{i\omega_n + ik_n - E_+(k+q/2)} \frac{1}{ik_n - E_+(k-q/2)} = \frac{n_f(k-q/2) - n_f(k+q/2)}{i\omega_n - (E_+(k+q/2) - E_+(k-q/2))}$$

For "+" band fully occupied, no Fermi surfaces, as $q \to 0$, but ω finite

$$\sum_{ik_n} \cdots = 0 \text{ in the } \lim_{\omega\to 0} \lim_{q \to 0}.$$

Similarly $$\sum_{ik_n} \frac{1}{i\omega_n + ik_n - E_-(k+q/2)} \frac{1}{ik_n - E_-(k-q/2)} = 0$$

in the $\lim_{\omega\to 0} \lim_{q\to 0}$.

- As for inter-band transitions, we need a few trace identities

  - $tr[(1 \pm \vec{\sigma}\cdot\hat{d})(1 \mp \vec{\sigma}\cdot\hat{d})] = 0$

  - $tr[(1 \pm \vec{\sigma}\cdot\hat{d})\sigma^\alpha(1 \mp \vec{\sigma}\cdot\hat{d})] = \pm tr[\vec{\sigma}\cdot\hat{d}\,\sigma^\alpha] \mp tr[\sigma^\alpha \vec{\sigma}\cdot\hat{d}]$

    $- tr[\sigma^\gamma d^\gamma \sigma^\alpha \sigma^\beta d^\beta] = -d^\beta d^\gamma \, tr[\sigma^\alpha\sigma^\beta\sigma^\gamma] = -2i\epsilon^{\alpha\beta\gamma} d^\beta$

  $\Rightarrow tr[(1 \pm \vec{\sigma}\cdot\hat{d})\sigma^\alpha(1 \mp \vec{\sigma}\cdot\hat{d})] = 0$, similarly

  $tr[(1 \pm \vec{\sigma}\cdot\hat{d})(1 \mp \vec{\sigma}\cdot\hat{d})\sigma^\beta] = 0$

  - $tr[(1 \pm \vec{\sigma}\cdot\hat{d})\sigma^\alpha(1 \mp \vec{\sigma}\cdot\hat{d})\sigma^\beta] = tr[\sigma^\alpha\sigma^\beta \pm \vec{\sigma}\cdot\hat{d}\,\sigma^\alpha\sigma^\beta \mp \sigma^\alpha\vec{\sigma}\cdot\hat{d}\,\sigma^\beta - (\vec{\sigma}\cdot\hat{d})\sigma^\alpha(\vec{\sigma}\cdot\hat{d})\sigma^\beta]$

$= 2\delta^{\alpha\beta} \pm 2i\epsilon^{\alpha\beta\gamma} d^\gamma \mp 2i\epsilon^{\alpha\gamma\beta} d^\gamma - [2d^\alpha d^\beta + 2d^\beta d^\alpha - 2\hat{d}\cdot\hat{d}\,\delta^{\alpha\beta}]$

$= 4[\delta^{\alpha\beta} - \hat{d}^\alpha\hat{d}^\beta \pm i\epsilon^{\alpha\beta\gamma} d^\gamma]$

inter band transition

$$\mathrm{tr}\left[(1\pm\vec{\sigma}\cdot\hat{d})\left(\frac{\partial \varepsilon}{\partial k_i}+\frac{\partial d^\alpha}{\partial k_i}\sigma^\alpha\right)(1\mp\vec{\sigma}\cdot\hat{d})\left(\frac{\partial \varepsilon}{\partial k_j}+\frac{\partial d^\beta}{\partial k_j}\sigma^\beta\right)\right]$$

$$=\frac{\partial \varepsilon}{\partial k_i}\frac{\partial \varepsilon}{\partial k_j}\mathrm{tr}\big[\underbrace{(1\pm\vec{\sigma}\cdot\hat{d})(1\mp\vec{\sigma}\cdot\hat{d})}_{1}\big]+\frac{\partial d_\alpha}{\partial k_i}\frac{\partial \varepsilon}{\partial k_j}\mathrm{tr}\big[\underbrace{(1\pm\vec{\sigma}\cdot\hat{d})\sigma^\alpha(1\mp\vec{\sigma}\cdot\hat{d})}_{2}\big]$$

$$+\frac{\partial \varepsilon}{\partial k_i}\frac{\partial d^\beta}{\partial k_j}\mathrm{tr}\big[\underbrace{(1\pm\vec{\sigma}\cdot\hat{d})(1\mp\vec{\sigma}\cdot\hat{d})\sigma^\beta}_{3}\big]+\frac{\partial d_\alpha}{\partial k_i}\frac{\partial d_\beta}{\partial k_j}\mathrm{tr}\big[\underbrace{(1\pm\vec{\sigma}\hat{d})\sigma^\alpha(1\mp\vec{\sigma}\cdot\hat{d})\sigma^\beta}_{4}\big]$$

$1, 2, 3 = 0$

$$=4\frac{\partial d_\alpha}{\partial k_i}\frac{\partial d_\beta}{\partial k_j}\left(\delta_{\alpha\beta}-\hat{d}_\alpha\hat{d}_\beta\pm i\epsilon_{\alpha\beta\gamma}\hat{d}_\gamma\right)$$

The first two terms $\quad d_\alpha = d\,\hat{d}_\alpha \quad$ where $d=|\vec{d}|$

$$\frac{\partial d_\alpha}{\partial k_i}=\frac{\partial d}{\partial k_i}\hat{d}_\alpha+d\frac{\partial}{\partial k_i}\hat{d}_\alpha$$

$$\Rightarrow \frac{\partial d_\alpha}{\partial k_i}\frac{\partial d_\beta}{\partial k_j}(\delta_{\alpha\beta}-\hat{d}_\alpha\hat{d}_\beta)=\Big[\frac{\partial d}{\partial k_i}\hat{d}_\alpha\frac{\partial d}{\partial k_j}\hat{d}_\beta+\frac{\partial d}{\partial k_i}\hat{d}_\alpha\, d\frac{\partial}{\partial k_j}\hat{d}_\beta$$

$$+d\frac{\partial}{\partial k_i}\hat{d}_\alpha\frac{\partial d}{\partial k_j}\hat{d}_\beta+d^2\frac{\partial}{\partial k_i}\hat{d}_\alpha\cdot\frac{\partial}{\partial k_j}\hat{d}_\beta\Big]\left[\delta_{\alpha\beta}-\hat{d}_\alpha\hat{d}_\beta\right]$$

$$=\frac{\partial d}{\partial k_i}\frac{\partial d}{\partial k_j}+d\frac{\partial d}{\partial k_i}\underbrace{\hat{d}\cdot\frac{\partial}{\partial k_j}\hat{d}}_{=0}+d\frac{\partial d}{\partial k_j}\underbrace{\left(\frac{\partial}{\partial k_i}\hat{d}\right)\cdot\hat{d}}_{=0}+d^2\frac{\partial}{\partial k_i}\hat{d}\cdot\frac{\partial}{\partial k_j}\hat{d}$$

$$-\frac{\partial d}{\partial k_i}\frac{\partial d}{\partial k_j}-d\frac{\partial d}{\partial k_i}\underbrace{\hat{d}\frac{\partial}{\partial k_j}\hat{d}}_{=0}-\underbrace{\hat{d}\frac{\partial}{\partial k_i}\hat{d}}_{=0}\,d\frac{\partial d}{\partial k_j}-d^2\underbrace{\left(\hat{d}\frac{\partial}{\partial k_i}\hat{d}\right)\left(\hat{d}\frac{\partial}{\partial k_j}\hat{d}\right)}_{=0}$$

$$=d^2\frac{\partial}{\partial k_i}\hat{d}\cdot\frac{\partial}{\partial k_j}\hat{d}$$

$$\epsilon_{\alpha\beta\gamma}\frac{\partial d_\alpha}{\partial k_i}\frac{\partial d_\beta}{\partial k_j}\hat{d}_\gamma = \epsilon_{\alpha\beta\gamma}\left[\frac{\partial d}{\partial k_i}\hat{d}_\alpha + d\frac{\partial}{\partial k_i}\hat{d}_\alpha\right]\left[\frac{\partial d}{\partial k_j}\hat{d}_\beta + d\frac{\partial}{\partial k_j}\hat{d}_\beta\right]\hat{d}_\gamma$$

$$= \epsilon_{\alpha\beta\gamma}\left[\frac{\partial d}{\partial k_i}\frac{\partial d}{\partial k_j}\underline{\hat{d}_\alpha\hat{d}_\beta\hat{d}_\gamma} + d\frac{\partial d}{\partial k_i}\underline{\underline{\hat{d}_\alpha}}\left(\frac{\partial}{\partial k_j}\hat{d}_\beta\right)\underline{\underline{\hat{d}_\gamma}} + d\left(\frac{\partial}{\partial k_i}\hat{d}_\alpha\right)\frac{\partial d}{\partial k_j}\underline{\underline{\hat{d}_\beta\hat{d}_\gamma}}\right]$$

$$+\epsilon_{\alpha\beta\gamma}\, d^2\left(\frac{\partial}{\partial k_i}\hat{d}_\alpha\right)\left(\frac{\partial}{\partial k_j}\hat{d}_\beta\right)\hat{d}_\gamma$$

$$Q^{(1)}_{ij}(q=0, i\omega_n) = \frac{1}{4V\beta}\sum_{k, ik_n}\underbrace{\frac{\partial d_\alpha}{\partial k_i}\frac{\partial d_\beta}{\partial k_j}d_\gamma}_{4i\epsilon_{\alpha\beta\gamma}}\left\{\frac{1}{i\omega_n+ik_n-E_+(k)}\frac{1}{ik_n-E_-(k)} - \frac{1}{i\omega_n+ik_n-E_-(k)}\frac{1}{ik_n-E_+(k)}\right\}$$

$$= \frac{i\epsilon_{\alpha\beta\gamma}}{V}\sum_k \underbrace{d^2\left(\frac{\partial}{\partial k_i}\hat{d}_\alpha\right)\left(\frac{\partial}{\partial k_j}\hat{d}_\beta\right)\hat{d}_\gamma}_{\epsilon_{\alpha\beta\gamma}}\left[\frac{n_f(E_-(k))-n_f(E_+(k))}{i\omega_n+E_-(k)-E_+(k)} - \frac{n_f(E_+)-n_f(E_-)}{i\omega_n+E_+-E_-}\right]$$

since $n_f(E_+)=0$, $n_f(E_-)=1$

$$E_+(k)-E_- = 2d(k) \Rightarrow \frac{1}{i\omega_n-2d}+\frac{1}{i\omega_n+2d}$$

$$\xrightarrow{i\omega\to\omega+i\eta} = \frac{2\omega}{+\omega^2-(4d^2)} \to \frac{-\omega}{2d^2}$$

$$\Rightarrow Q^{(1)}_{ij}(q=0,\omega\to 0) = \frac{-i\omega}{2V}\sum_{\vec{k}}\epsilon_{\alpha\beta\gamma}\frac{\partial\hat{d}_\alpha}{\partial k_i}\frac{\partial\hat{d}_\beta}{\partial k_j}\hat{d}_\gamma$$

→ this part contribute ($Q^{(1)}_{ij} = -Q^{(1)}_{ji}$ ← contribute to Hall conductance)

$$\sigma_{ij} = \lim_{\omega\to 0}\frac{ie^2}{\omega\hbar}Q_{ij} = \frac{e^2}{2\hbar}\int\frac{dk_x dk_y}{4\pi^2}\left(\frac{\partial\vec{d}}{\partial k_x}\times\frac{\partial\vec{d}}{\partial k_y}\right)\cdot\hat{d}_r = \boxed{\frac{e^2}{h}\int\frac{dk_x dk_y}{8\pi}\cdot \epsilon_{ij}\epsilon_{\alpha\beta\gamma}\frac{\partial d_\alpha}{\partial k_i}\frac{\partial d_\beta}{\partial k_j}d_\gamma}$$

Then how about the contribution from $d^2 \frac{\partial}{\partial k_i}\hat{d}\cdot\frac{\partial}{\partial k_j}\hat{d}$

$$Q^{(2)}_{ij}(q=0, i\omega_n) = \frac{1}{V\beta}\sum_{k, ik_n} d^2 \frac{\partial}{\partial k_i}\hat{d}\cdot\frac{\partial}{\partial k_j}\hat{d}\left\{\frac{1}{i\omega_n + ik_n - E_+(k)}\,\frac{1}{ik_n - E_-(k)} + \frac{1}{i\omega_n + ik_n - E_-(k)}\,\frac{1}{ik_n - E_+(k)}\right\}$$

$$= \frac{1}{V}\sum_{\vec{k}} d^2 \frac{\partial}{\partial k_i}\hat{d}\cdot\frac{\partial}{\partial k_j}\hat{d}\left\{\frac{n_f(E_-(k)) - n_f(E_+(k))}{i\omega_n + E_-(k) - E_+(k)} + \frac{n_f(E_+(k)) - n_f(E_-(k))}{i\omega_n + E_+(k) - E_-(k)}\right\}$$

$$\xrightarrow[T=0\,K]{i\omega \to \omega + i\eta} Q^{(2)}_{ij}(q=0, \omega + i\eta) = \frac{1}{V}\sum_{\vec{k}} d^2 \frac{\partial}{\partial k_i}\hat{d}\cdot\frac{\partial}{\partial k_j}\hat{d}\left[\underbrace{\frac{1}{\omega - 2|d|} - \frac{1}{\omega + 2|d|}}_{\frac{|d|}{(\frac{\omega}{2})^2 - d^2}}\right]$$

$$\boxed{\omega \to 0 \Rightarrow Q^{(2)}_{ij}(q=0, \omega + i\eta) = \iint \frac{dk_x dk_y}{(2\pi)^2}\, d(\vec{k})\, \frac{\partial}{\partial k_i}\hat{d}\cdot\frac{\partial}{\partial k_j}\hat{d}}$$

$Q^{(2)}_{ij}$ is symmetric with respect to i, j $\Rightarrow$ it's not Hall conductance. But

$$\vec{j} = \frac{\partial}{\partial t}\vec{P} \Rightarrow j_i(\omega) = -i\omega P_i(\omega)$$

$$\vec{E} = -\frac{1}{c}\frac{\partial}{\partial t}\vec{A} \qquad E_i(\omega) = +\frac{i\omega}{c}A_i(\omega)$$

$$\Rightarrow j_i(\omega) = Q^{(2)}_{ij} A_j(\omega)$$

$$\Rightarrow -i\omega P_i(\omega) = Q^{(2)}_{ij}\frac{c}{i\omega}E_j(\omega) \Rightarrow P_i(\omega) = \frac{c}{\omega^2}Q^{(2)}_{ij}E_j(\omega)$$

$$\Rightarrow \chi_{ij} = \frac{c}{\omega^2}Q^{(2)}_{ij}$$ ← related to polarization?

How about for QAHE metal? We need to check that the intra-band transition still does not contribute to σ_{xy}. even with Fermi surface.

we use trace identity: $\quad tr[(1\pm\vec{\sigma}\cdot\hat{d})\sigma^\alpha(1\pm\vec{\sigma}\cdot\hat{d})\sigma^\beta] = 4\hat{d}^\alpha\hat{d}^\beta$

$$tr[(1\pm\vec{\sigma}\cdot\hat{d})(1\pm\vec{\sigma}\cdot\hat{d})] = tr[2(1\pm\vec{\sigma}\cdot\hat{d})] = 4$$

$$tr[(1\pm\vec{\sigma}\cdot\hat{d})\sigma^\alpha(1\pm\vec{\sigma}\cdot\hat{d})] =$$

$$= tr[(1\pm\vec{\sigma}\cdot\hat{d})^2\sigma^\alpha] = 2\,tr[(1\pm\vec{\sigma}\cdot\hat{d})\sigma^\alpha] = \pm 4\hat{d}^\alpha$$

$$tr\left[(1\pm\vec{\sigma}\cdot\hat{d})\left[\frac{\partial\varepsilon}{\partial k_i} + \frac{\partial d^\alpha}{\partial k_i}\sigma^\alpha\right](1\pm\vec{\sigma}\cdot\hat{d})\left[\frac{\partial\varepsilon}{\partial k_j} + \frac{\partial d^\beta}{\partial k_j}\sigma^\beta\right]\right]$$

$$= \frac{\partial\varepsilon}{\partial k_i}\frac{\partial\varepsilon}{\partial k_j}\,tr[(1\pm\vec{\sigma}\cdot\hat{d})^2] + \frac{\partial d^\alpha}{\partial k_i}\frac{\partial\varepsilon}{\partial k_j}\,tr[(1\pm\vec{\sigma}\cdot\hat{d})\sigma^\alpha(1\pm\vec{\sigma}\cdot\hat{d})]$$

$$+ \frac{\partial\varepsilon}{\partial k_i}\frac{\partial d^\beta}{\partial k_j}\,tr[(1\pm\vec{\sigma}\cdot\hat{d})(1\pm\vec{\sigma}\cdot\hat{d})\sigma^\beta]$$

$$+ \frac{\partial d^\alpha}{\partial k_i}\frac{\partial d^\beta}{\partial k_j}\,tr[(1\pm\vec{\sigma}\cdot\hat{d})\sigma^\alpha(1\pm\vec{\sigma}\cdot\hat{d})\sigma^\beta]$$

$$= 4\frac{\partial\varepsilon}{\partial k_i}\frac{\partial\varepsilon}{\partial k_j} \pm 4\hat{d}^\alpha\frac{\partial d^\alpha}{\partial k_i}\frac{\partial\varepsilon}{\partial k_j} \pm 4\hat{d}^\beta\frac{\partial\varepsilon}{\partial k_i}\frac{\partial d^\beta}{\partial k_j} + 4\frac{\partial d^\alpha}{\partial k_i}\hat{d}^\alpha\frac{\partial d^\beta}{\partial k_j}\hat{d}^\beta$$

$$= 4\left[\frac{\partial\varepsilon}{\partial k_i}\frac{\partial\varepsilon}{\partial k_j} \pm \left(\frac{\partial d}{\partial k_i}\frac{\partial\varepsilon}{\partial k_j} + \frac{\partial d}{\partial k_j}\frac{\partial\varepsilon}{\partial k_i}\right) + \frac{\partial d}{\partial k_i}\frac{\partial d}{\partial k_j}\right]$$

$\Rightarrow$ all the terms are symmetric with respect to $(ij \Leftrightarrow ji)$

$\Rightarrow$ all the intra-band transition doesn't ~~to~~ contribute to σ_{xy}.

$$\Rightarrow \sigma_{ij} = \frac{e^2}{h} \frac{1}{8\pi} \oiint d^2\vec{k} \; \epsilon_{ij} \left(\frac{\partial \vec{d}}{\partial k_i} \times \frac{\partial \vec{d}}{\partial k_j} \right) \cdot \vec{d}$$ ← for insulators

Chern number → Pontragian index

complex tangent bundle