"Fate of the false vacuum"

We have studied 0d field theory — QM. Now we move to real field theory, at each $(\vec{x}, t)$, there exists a field variable $\phi(\vec{x}, t)$. Here $\vec{x}$ is just an index, which place the role of t of $x(t)$ in QM.

$\vec{x}(t)$

$\downarrow \quad \downarrow$

$\phi(x, t)$

Let's write down the Hamiltonian density

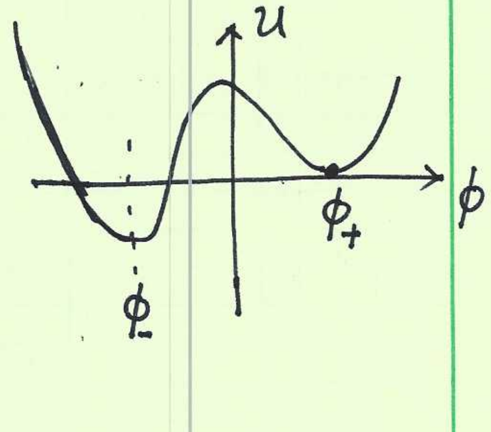
$$\begin{cases} \hat{H}(\vec{x}) = \dfrac{\hat{\pi}^2}{2m} + \dfrac{k_s a^2}{2}(\partial_x \hat{\phi})^2 + U(\phi) \\ [\hat{\phi}(x), \hat{\pi}(x')] = i\hbar \delta(x - x') \end{cases}$$

$$\rightarrow \begin{cases} \partial_t \phi = \dfrac{\partial \hat{H}}{\partial \pi} = \dfrac{\hat{\pi}}{m} \\ \mathcal{L} = \hat{\pi} \partial_t \phi - H \end{cases} \Rightarrow \mathcal{L} = \frac{m}{2}(\partial_t \phi)^2 - \frac{k_s}{2} a^2 (\partial_x \hat{\phi})^2 - U(\phi)$$

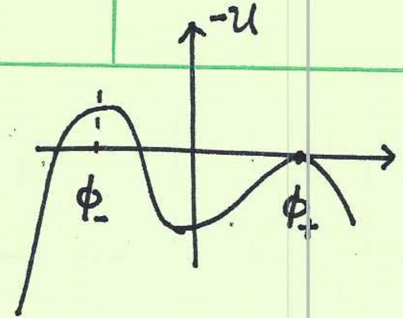
change to Euclidean formalism, and simplify $\mathcal{L}_E = \frac{1}{2}(\partial_\mu \phi)^2 + U(\phi)$. (we absorb m, $k_s a$ into $\hat{\phi}$ and rescaling t, and x), then $\mathcal{L}_E$ has 4d rotational symmetry.

$$S[\phi] = \int_0^T d\tau \int_0^L d^3\vec{x} \; \frac{1}{2}(\partial_\mu \phi)^2 + U(\phi) \quad \text{and} \quad Z = \int D\phi(x, \tau)\, e^{-S(\phi)}$$

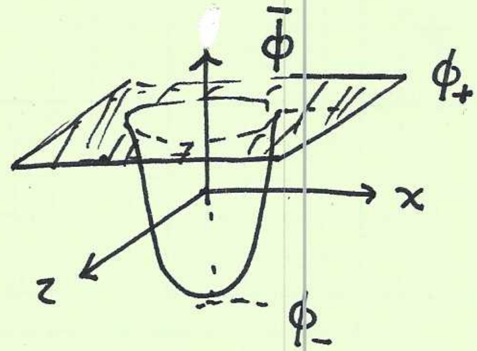
Suppose at $\tau = 0$, the system stays in the false vacuum ϕ_+, we can calculate the survival probability of the metastable state.



The bounce solution:



Consider a finite volume $\phi(\vec{x})$ fluctuate to ϕ_- and then go back.



The saddle point Eq

$$\delta S = \int d^4x \left[-\delta\phi\, \partial_\mu^2 \phi + \delta\phi\, \mathcal{U}'(\phi) \right] = 0$$

$$\Rightarrow \quad \partial_\mu \partial_\mu \bar{\Phi} = \mathcal{U}'(\bar{\Phi})$$

We want solutions with finite action $\left\{ \begin{array}{l} \lim\limits_{\tau\to\infty} \bar{\Phi}(\vec{x}, x_4) = \phi_+ \\ \lim\limits_{|\vec{x}|\to\infty} \bar{\Phi}(\vec{x}, x_4) = \phi_+ \end{array} \right.$

Once we have found a bounce solution with the action $S_0 = S(\bar{\Phi})$, we have $\Gamma/V = K\, e^{-S_0}$, where K is a determinantal factor.

Remark ① The power of the method: the decay problem of infinite degrees of freedom $\longrightarrow$ a classic solution of PDE.

② K carries the unit of [energy]/[time][volume].
The bounce solution breaks time and spatial translation symmety: the zero mode contribute $T \cdot V$ factor.

③ We are not interested the trivial solution $\phi = \phi_+$, which can change the real part of E_G, but no negative eigenvalues $\rightarrow$ no contribution to vacuum decay!

④ Suppose we have solved $\bar{\Phi}(x)$, and now we write $\phi_\lambda(x) = \bar{\Phi}(x/\lambda)$

and the action of $\phi_\lambda \Rightarrow S(\lambda) = \int d^4x \; \frac{1}{2}\left(\partial_\mu \bar{\Phi}(x/\lambda)\right)^2 + u(\bar{\phi}(x/\lambda))$

$$= \lambda^2 \int d^4x' \; \frac{1}{2}\left(\partial'_\mu \bar{\phi}(x')\right)^2 + \lambda^4 \int d^4x' \, u(\bar{\phi}(x'))$$

$\lambda = 1 \Rightarrow$ classic orbit $\Rightarrow \left.\frac{dS(\lambda)}{d\lambda}\right|_{\lambda=1} = 0 \Rightarrow \frac{2\lambda}{2}\int d^4x \left(\partial_\mu \bar{\Phi}\right)^2 + 4\lambda^3 \int d^4x \, u(\bar{\Phi}) = 0$

↗ set $\lambda = 1$

$$\Rightarrow \int d^4x \, u(\bar{\Phi}) = -\frac{1}{4}\int d^4x \left(\partial_\mu \bar{\phi}\right)^2$$

$$\Rightarrow \boxed{S_0 = \frac{1}{4}\int d^4x \left(\partial_\mu \bar{\Phi}\right)^2 > 0}$$

although u is negative during the bounce, but S_0 remains positive.

but $\left.\frac{d^2S}{d\lambda^2}\right|_{\lambda=1} = \int d^4x \left[\left(\partial_\mu \bar{\phi}\right)^2 + 12 \, u(\bar{\phi})\right] = -2\int d^4x \left(\partial_\mu \bar{\phi}\right)^2 < 0$

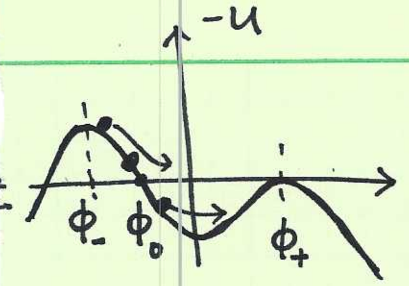
So by varying λ, S_0 can be lowered, thus $\bar{\Phi}$ cannot be the one with lowerest energy. There must a negative eigenvalue solution.

★(if for $d=1$, we'll have $\frac{d^2S}{d\lambda^2} > 0$, so the 1D instanton can be the lowest energy. — more comments later).

Now we consider $O(4)$ solution: $\bar{\Phi}(r)$ satisfies,

$$\frac{d^2}{dr^2}\bar{\Phi} + \frac{3}{r}\frac{d\bar{\Phi}}{dr} = u'(\bar{\Phi}), \quad \text{with } \lim_{r\to\infty} \bar{\Phi}(r) = \phi_+ \text{ and } \left.\frac{d\bar{\phi}}{dr}\right|_{r=0} = 0.$$

we can interprete r as time t. Eq describes a particle in the potential $-U$ subject to the viscoucity force. The particle is released at rest at time zero.



If at $r=0$, if we ~~start~~ with ϕ_-, although there exists friction, we can show it will overshoot.

If we ~~start~~ on the very right to ϕ_-, then we will have undershoot. — even with the friction

Now justify the overshooting issue: suppose ϕ initially very close to ϕ_-, then $u(\phi) = +\frac{1}{2}u''(\phi_-)(\phi-\phi_-)^2$, and we safely linearlize the Eq

$$\left[\frac{d^2}{dr^2} + \frac{3}{r} - u''(\phi_-)\right](\bar{\phi}(r) - \phi_-) = 0$$

$$\Rightarrow \quad \bar{\phi}(r) - \phi_- = 2(\bar{\phi}(0) - \phi_-)\frac{I_1(\mu r)}{\mu r}, \quad \text{where } \mu^2 = u''(\phi_-)$$

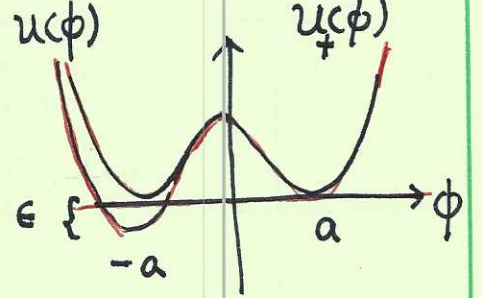
$I_1(\mu r)$ is imaginary argument Bessel function and $I_1(\mu r) \xrightarrow[\text{as } r\to 0]{} \mu r$.

So as long as $\bar{\phi}(0) \to \phi_-$, we can have $\bar{\phi}(r)$ sufficiently approaches ϕ_-, even though at large distance of r. And as r goes large, the friction goes small. So as long as $u(\phi_-) < 0$, it's always overshoot. So the right-shoot is a little tricky. Its poisition $u(\phi_0)$ is atually < 0, some place right to ϕ_- but still negative. This is the reason that the classic solution $\bar{\phi}$ has

$\int d^4x \, u(\bar{\phi}) < 0$. This is different from the 0+1 d case in which there's no friction.

- Now we need to calculate S_0 by assuming $\mathcal{U}(\phi_+) - \mathcal{U}(\phi_-)$ is small.

To be concrete, we define $\mathcal{U}_+(\phi)$ as a symmetrized function with the same minimum at $\phi = \pm a$, and define

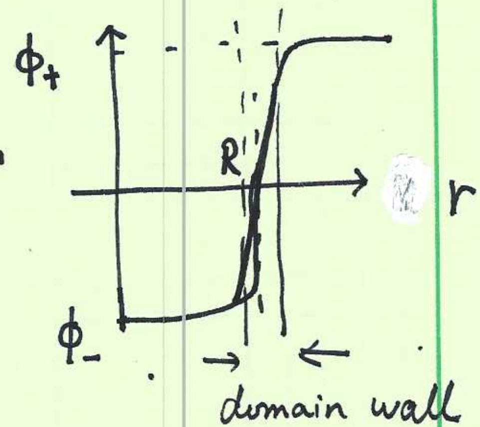


$$\begin{cases} \mathcal{U} = \mathcal{U}_+ + \epsilon(\phi - a)/2a \\ \phi_\pm = \pm a \end{cases}$$

where ϵ is energy difference between the true and false vacua.

We seek the solution: $\bar{\Phi}(r)$ very close to ϕ_- for $r < R$, and near $r = R$, ϕ_- quickly changes from $\phi_- \to \phi_+$, and move slowly as $r \gg R$.

Around the domain region, we neglet the viscosity term, and neglect the difference between $\mathcal{U}_+$ and $\mathcal{U}$.



$$\Rightarrow \quad d^2\bar{\Phi}/dr^2 = \mathcal{U}_+'(\bar{\Phi})$$

This is solution of 0+1d, instanton solution. We evaluate

$$S = 2\pi^2 \int_0^\infty r^3 dr \left[\frac{1}{2} \left(d\bar{\Phi}/dr \right)^2 + \mathcal{U} \right]$$

① Outside the bubble, $\mathcal{U}(\phi_+) = 0$, and $d\bar{\Phi}/dr = 0$ $\to$ no contribution

② inside the bubble $\mathcal{U}(\phi_-) = -\epsilon \Rightarrow -\frac{1}{2}\pi^2 R^4 \epsilon$

③ domain wall contribution $r \simeq R$

$$\Delta S = 2\pi^2 R^3 \int_{R-\Delta r}^{R+\Delta r} dr \, \frac{1}{2} \left(\frac{d\bar{\Phi}}{dr} \right)^2 + \mathcal{U}_+$$

In this region, $\left(\frac{d\bar{\Phi}}{dr}\right)^2 - 2U_+ = \text{const} = 0$

$$\Rightarrow \Delta S = 2\pi^2 R^3 \int dr \left(\frac{d\bar{\Phi}}{dr}\right)^2 = 2\pi^2 R^3 \int d\bar{\Phi} \left(\frac{d\bar{\Phi}}{dr}\right)$$

$$= 2\pi^2 R^3 \int_{-a}^{a} d\bar{\Phi}\ \sqrt{2U_+} = 2\pi^2 R^3 S_1$$

↑ define as S_1

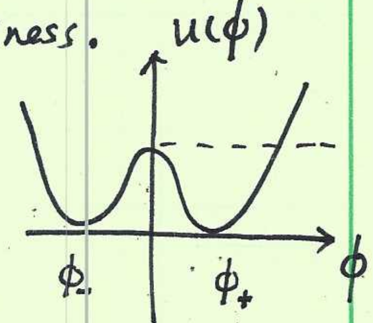
$$\Rightarrow S_{tot} = -\frac{1}{2}\pi^2 R^4 \epsilon + 2\pi^2 R^3 S_1 \qquad \Rightarrow \frac{dS}{dR} = 0 = -2\pi^2 R^3 \epsilon + 6\pi^2 R^2 S_1$$

$$R = 3S_1/\epsilon$$

$$\Rightarrow \boxed{S_0 = 27\pi^2 S_1^4 / 2\epsilon^3}$$

The above approximation is valid at small ϵ,

or the bubble size $R \gg$ the domian wall thickness.



as the bubble edge to middle of the bubble

$R \to R + \Delta r/2$, $\Rightarrow U'' \phi_-^2 = U(0)$

$\phi_- \to 0$

$\left(\frac{d\phi}{dr}\right)^2 \simeq 2U(0)$ ← classic equation of motion

↙ largest slope

$$\Rightarrow U'' \phi_-^2 \simeq \left(\frac{d\phi}{dr}\right)^2 \simeq \left(\frac{|\phi_-|}{\Delta r}\right)^2 \Rightarrow$$ the domain wall $\Delta r \simeq (U'')^{-1/2}$

$\Rightarrow$ The above approx is valid at

$$\boxed{R = \frac{3S_1}{\epsilon} \gg \left[U''(\phi_-)\right]^{-1/2}}$$

← check unit

$[S_1] = [U]^{1/2}[\phi]$

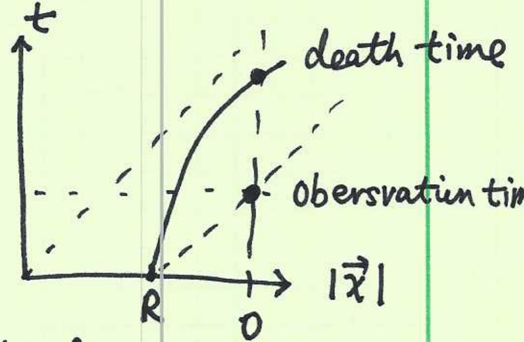
$[U''] = [U][\phi]^{-2}$

$\Rightarrow [\epsilon] = [U]$ correct!

§ The real-time picture — Minkowski space

$$(\nabla^2 - \partial_t^2)\,\bar{\phi} = \mathcal{U}'(\bar{\phi}) \;\Rightarrow\; \phi(\vec{x}, t) = \bar{\phi}\left(r = \sqrt{|x|^2 - t^2}\right)$$

☆ The bubble surface $|x|^2 - t^2 = R^2$. After $t > 0$, it becomes as real event. At $t < 0$, it's quantum fluctuation. $t = 0$ is called materalization.



☆ For an oberserer outside the surface, when he oberseve the bubble, and when he is absorbed in the bubble as shown in the figure.

☆ where $\underbrace{\ }_{\text{does}}$ the energy go?

The wall expasion velocity
$$v = \frac{d|x|}{dt} = \frac{d}{dt}(R^2 + t^2)^{1/2} = \frac{t}{\sqrt{R^2 + t^2}} = \frac{(x^2 - R^2)}{|x|}$$
$$= \left[1 - R^2/|x|^2\right]^{1/2}$$

~~this region is moving, due to length~~

Since the expasion follows the Lorentz transformation: The energy of a unit area on the bubble can be obtained from Lorentz transform on it's starting point. At $t = 0$, the bubble wall is at rest with $\underbrace{\text{energy } S_1}_{\text{density}}$.

According to [illegible] 4-vector $S'^2(v) - \frac{S'^2(v)}{c^2} v^2 = S_1^2$ ← static mass

$$\Rightarrow\; S'(v) = \frac{S_1}{\sqrt{1 - \left(\frac{v}{c}\right)^2}}$$

$$\Rightarrow\; E_{wall} = 4\pi |x|^2 S_1 / (1 - v^2)^{1/2} = \frac{4\pi |x|^3}{R} S_1 = \frac{4\pi}{3}|x|^3 \epsilon.$$

This means that all the energy realeased from the false vacuum is spent on the acceleration of the wall.

★ Other subtle points

① Zero modes — since our solution has 4 different translation symmetries along τ, x. y. z, there're 4 zero modes.

$$\delta\phi = C_0\phi_0 + C_1\phi_1 + C_2\phi_2 + C_3\phi_3 + \sum_{n>3}^{N} C_n\phi_n$$

$$KTL^3 = \int \frac{dC_0\,dC_1\,dC_2\,dC_3}{(\sqrt{2\pi\hbar})^4} \left\{ \frac{\mathrm{Det}'[-\partial_\mu^2 + \mathcal{U}''(\bar\phi)]}{\mathrm{Det}[-\partial_\mu^2 + \mathcal{U}''(\phi_+)]} \right\}^{-1/2}$$

$$S_0 = \int d^4x\,(\partial_\mu\bar\phi)^2$$, because isotropy

$$\int d^4x\,(\partial_i\bar\phi)^2 = \frac{S_0}{4}$$ ⇒ the normalized zero mode

$$\int d^4x \left(\frac{\partial_x\bar\phi}{\sqrt{S_0/4}}\right)^2 = \cdots = \int d^4x \left(\frac{\partial_0\bar\phi}{\sqrt{S_0/4}}\right)^2 = 1$$

$$\Rightarrow \delta\phi = \frac{C_0}{\sqrt{S_0/4}}(\partial_0\bar\phi) + \frac{C_1}{\sqrt{S_0/4}}(\partial_x\bar\phi) + \cdots$$

the shift $\Delta\tau \in [-T/2, T/2]$, $\Delta x \in [-L/2, L/2]$

$$\int dC_0 \frac{1}{\sqrt{S_0/4}} = \int d\tau\ , \quad \int dC_1 \frac{1}{\sqrt{S_0/4}} = \int dx$$

$$\Rightarrow \int dC_0 \cdots dC_3 = \int d\tau\,dx\,dy\,dz \left(\frac{S_0}{4}\right)^2$$

$$\Rightarrow K T L^3 = T L^3 \left(\frac{\frac{S_0}{2}}{2\pi\hbar} \right)^{4/2} \left\{ \frac{\mathrm{Det}'(-\partial_\mu^2 + U''(\bar{\phi}))}{\mathrm{Det}(-\partial_\mu^2 + U''(\phi_+))} \right\}^{-1/2}$$

$$K = \frac{S_0^2}{16\pi^2\hbar^2} \left[\frac{\mathrm{Det}'(-\partial_\mu^2 + U''(\bar{\phi}))}{\mathrm{Det}(-\partial_\mu^2 + U''(\phi_+))} \right]^{-1/2}$$

K's unit : $\frac{1}{T \cdot V}$

$$\boxed{\Gamma/V = K e^{-S_0}}$$

2: Negative eigenvalue issue :

it can be proved that there's only one negative eigenvalue.