Bethe ansatz — spin chain

§ wavefunction

$$\boxed{\psi(x_1, \cdots x_M) = \sum_P A_P \, e^{i(k_{P_1} x_1 + k_{P_2} x_2 + \cdots + k_{P_M} x_M)}}$$

Conside 2 permutations connected by a neareast neighbour exchange

$$P: (k_{P_1} \cdots k_{P_j} = k, \; k_{P_{j+1}} = k', \cdots) \leftarrow A_P$$

$$P: (k_{P'_1} \cdots k_{P'_j} = k', \; k_{P'_{j+1}} = k, \cdots) \leftarrow A_{P'}$$

$$\boxed{A_{P'}/A_P \equiv - e^{i\theta(k', k)} = - \frac{e^{i(k+k')} - 2\Delta e^{ik'} + 1}{e^{i(k+k')} - 2\Delta e^{ik} + 1}}$$

The BA equations

$$\boxed{e^{ik_j N} = (-)^{M-1} e^{i \sum_{\ell=1}^{M} \theta(k_j, k_\ell)}} \quad j = 1, 2, \cdots M$$

and $E = J \sum_{j=1}^{M} (\cos k_j - \Delta)$

For $\Delta = 1$, parameterization

$$e^{ik_j} = \frac{\lambda_j + i/2}{\lambda_j - i/2} \Rightarrow \lambda_j = \frac{1}{2} \cot \frac{k_j}{2}$$

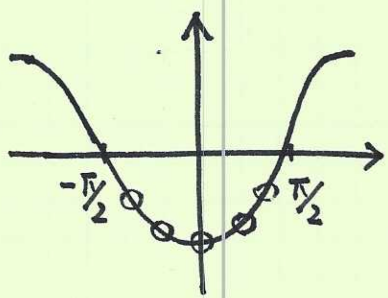
then $e^{i\theta(k_j, k_\ell)} = - \dfrac{\lambda_j - \lambda_\ell + i}{\lambda_j - \lambda_\ell - i}$

§ XY limit, and set $J \to -J$ (ferro XY model, $\Delta = 0$)

$$e^{ik_j N} = (-)^{M-1} \Rightarrow k_j = \frac{2\pi}{N} Q_j$$

Q_j: integer for odd M but half-integer for even M

$$E = -J \sum_{j=1}^{M} \cos k_j$$



For the ground state

$$Q_j = -\frac{M-1}{2}, -\frac{M-3}{2}, \cdots \frac{M-1}{2}$$

If M is odd, $k=0$ is included

If M is even, $k=0$ is not included.

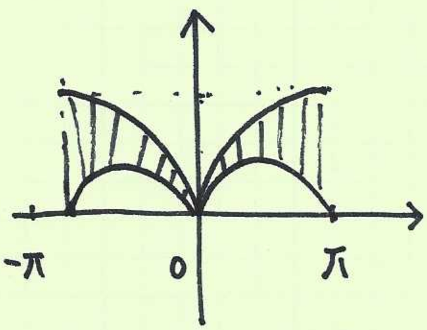
① Excitation spectrum: p-h (particle # conserved)

$$E(q) = -J\,[\cos(k+q) - \cos k] = 2J \sin(k + q/2) \sin q/2$$

The upper bound at $k = \frac{\pi}{2} - q/2 \Rightarrow E_u(q) = 2J \sin q/2$

The lower bound $\frac{\pi}{2} - q < k < \pi/2 \Rightarrow \frac{\pi}{2} - q/2 < k + q/2 < \frac{\pi}{2} + \frac{q}{2}$

$$\sin_{\min}(k + q/2) = \sin\left(\frac{\pi}{2} \mp \frac{q}{2}\right) = \cos\frac{q}{2} \Rightarrow E_L(q) = J|\sin q|.$$



2 consider S_+, then the ground state change particle number

$$\sum_{Q_j=-\frac{M-1}{2}}^{\frac{M-1}{2}} \cos\frac{2\pi}{N} Q_j = \sum_{Q_j=-\frac{M-1}{2}}^{\frac{M-1}{2}} e^{i\frac{2\pi}{N}Q_j}$$

$$= \frac{e^{-i\frac{2\pi}{N}\frac{M-1}{2}} - e^{i\frac{2\pi}{N}\frac{M+1}{2}}}{1-e^{i\frac{2\pi}{N}}} = \frac{e^{-i\frac{2\pi}{N}\frac{M}{2}} - e^{i\frac{2\pi}{N}\frac{M}{2}}}{e^{-i\pi/N}-e^{i\pi/N}}$$

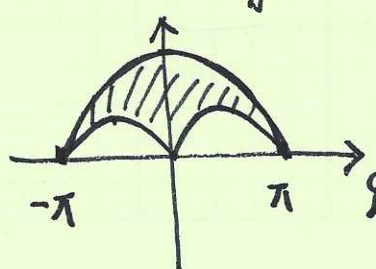
$$= \frac{\sin\left(\frac{M}{N}\pi\right)}{\sin(\pi/N)}$$

$\Rightarrow E_G(M) = -J\sin\frac{M}{N}\pi / \sin\frac{\pi}{N}$. At zero field, $M=\frac{N}{2}$, $E_G(M)$ reaches the lowest value. But as M changes from $\frac{N}{2}\to\frac{N}{2}\pm1$, the $E_G(\frac{N}{2}\pm1)$ $=-J\sin(\frac{1}{2}\pm\frac{1}{N})\pi/\sin\frac{\pi}{N} \Rightarrow E_G(\frac{N}{2}\pm1)-E_G(\frac{N}{2}) = \frac{J}{\sin\frac{\pi}{N}}\left[1-\cos\frac{\pi}{N}\right]$

$$\longrightarrow \frac{J}{2}\left(\frac{\pi}{N}\right) \to 0 \text{ as } N\to\infty.$$

So the extra p-h excitation based on $|\frac{N}{2}\pm1\rangle_G$ has the same shape as before. But for the AFM XY chain, the spectra minimal is located at $k=\pi$. If M is odd, the state of $k=\pi$ is occupied $\Rightarrow$ the total momenta of the system is π. If M is even, $\Rightarrow$ the total momenta is zero.

To count this effect, the $S_\pm$ spectra is shifted by π.



the upper boundary $2J\cos q/2$

the lower one $J|\sin q|$.

2 consider the case in the external field h with polarization

$$H = -J\sum_i (S_{x,i} S_{x,i+1} + S_{y,i} S_{y,i+1}) - h \sum_i S_{z,i}$$

For the case that the $|G\rangle$ has M down spin: $M < N/2$, then the total $S_z = \{(N-M) - M\}/2 = \frac{N}{2} - M$. The condition between M and h is $\boxed{E_G(M-1) - h < E_G(M) < E_G(M+1) + h}$

or in the continuum limit $\boxed{h = -\frac{\partial E_G}{\partial M} = +J \frac{\cos \frac{M}{N}\pi}{\sin \pi/N} \frac{\pi}{N}} > 0$

for $M < \frac{N}{2}$.

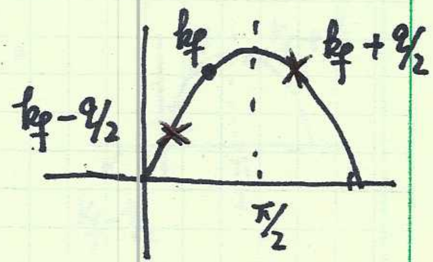
first consider particle-number conserved excitations

$$\boxed{k_f = \frac{2\pi}{N}\frac{M-1}{2} \sim \frac{\pi}{N} M} < \frac{\pi}{2}$$

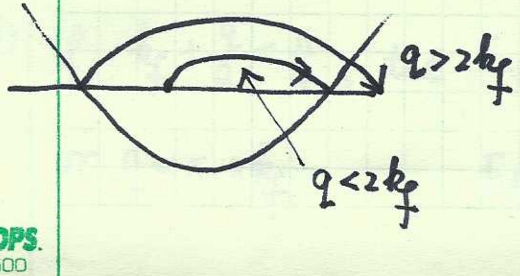
$E(q) = J[\cos k - \cos(k+q)]$ for $k_f - q < k < k_f \Rightarrow k_f - q/2 < k + q/2 < k_f + q/2$

1 Since $k_f < \pi/2$, the lower bound of $\sin(k + q/2)$ is taken at $k = k_f - q$

$$E_{lower}(q) = J[-\cos k_f + \cos(k_f - q)]$$
$$= 2J \sin(k_f - q/2) \sin q/2 \qquad \text{for } q < 2k_f$$



if $q > 2k_f$ $\quad E_{low}(q) = J[-\cos(-k_f + q) + \cos k_f]$



$\Rightarrow$ in total, we have

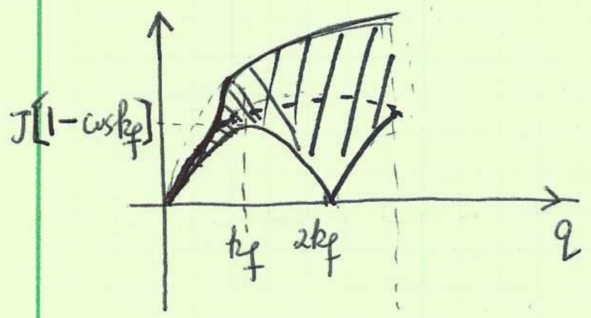
$$E_{low}(q) = 2J\,|\sin(k_f - q/2)|\,\sin\frac{q}{2}$$

② the upper bound. If $k_f + q/2 > \frac{\pi}{2}$, or $q > \pi - 2k_f$

$$E_{up}(q) = 2J \sin q/2 .$$

If $k_f + q/2 < \pi/2$ or $q < \pi - 2k_f$, $\Rightarrow E_{up} = J[\cos(k_f) - \cos(k_f + q)]$

$$= 2J \sin(k_f + q/2) \sin q/2$$

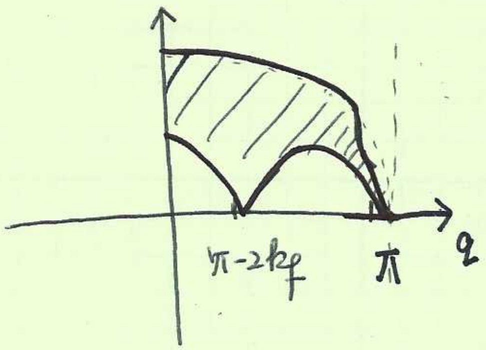


$$\Rightarrow E_{up} = \begin{cases} 2J \sin(k_f + q/2) \sin q/2 & \text{for } q < \pi - 2k_f \\ 2J \sin q/2 & \text{for } q > \pi - 2k_f . \end{cases}$$

③ for spin change $S_\pm$, $M \to M \pm 1$

$E_Q(M \mp 1) \pm h - E_G(M) = 0$. So when $-hS_z$ is counted in the Hamiltonian, the ground state energy shift goes to zero.

For the AFM, we need to shift the above p-h continuum by π



§ Heisenberg limit ($\Delta = 1$)

$$e^{ik_jN} = \prod_{\ell\neq j} \frac{\lambda_j - \lambda_\ell + i}{\lambda_j - \lambda_\ell - i} \qquad \text{where } \lambda_j = \tfrac{1}{2}\cot\frac{k_j}{2}$$

define $\phi_{j,\ell}$ as $e^{i\phi_{j,\ell}} = \dfrac{\lambda_j - \lambda_\ell + i}{\lambda_j - \lambda_\ell - i} \Rightarrow \lambda_j - \lambda_\ell = \cot\frac{1}{2}\phi_{j,\ell}$ $(j \neq \ell)$

$$\Rightarrow \begin{cases} N k_j = 2\pi m_j + \sum_{\ell\neq j} \phi_{j,\ell} \\ 2\cot\frac{1}{2}\phi_{j,\ell} = \cot\frac{k_j}{2} - \cot\frac{k_\ell}{2} \end{cases}$$

m_j's are integers mod N $\in \{1, 3, \cdots N-1\}$.

Then since $\phi_{j,\ell} = -\phi_{\ell,j} \Rightarrow \boxed{\sum_{j=1}^{M} k_j = \frac{2\pi}{N}\sum_{j=1}^{M} m_j} \to \left(\frac{N}{2}\right)^2$

$$\sum_{j=1}^{M} k = \frac{2\pi}{4}N = \pi N/2$$

★ Ground state pattern for $\{m_j\} = \{1, 3, \cdots N-1\}$

there're no ~~o~~ two m's adjacent to each other, — no bound states.

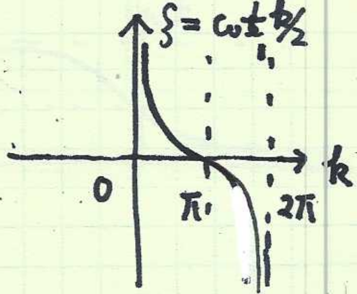
The key is to solve the distribution function of k. For this purpose, we define continous variables $x = \frac{2j-1}{N}$ and $y = \frac{2\ell-1}{N}$.

For $m_j = 2j-1$, $\to$ rescale to $0 < x < 1$. The Bethe Eq is reduced to

$$\boxed{\begin{gathered} k(x) = 2\pi x + \frac{1}{2}\int_0^1 dy\, \phi(k(x), k(y)), \text{ where} \\ \cot\frac{\phi(k(x), k(y))}{2} = \frac{1}{2}\cot\frac{k(x)}{2} - \frac{1}{2}\cot\frac{k(y)}{2} \\ \text{Then } E = J\sum_{j=1}^{N/2}(\cos k_j - 1) = \frac{N}{2}J\int_0^1 dx\left[\cos k(x) - 1\right] \end{gathered}}$$

$$\sum_j \to \int \frac{dx}{\frac{2}{N}} = \frac{N}{2}\int dx, \quad \sum_\ell \to \frac{N}{2}\int dy$$

Set $\xi = \cot \frac{k(x)}{2}$



as $\begin{cases} k: & 0 \to \pi \to 2\pi \\ \xi & +\infty \to 0 \to -\infty . \end{cases}$

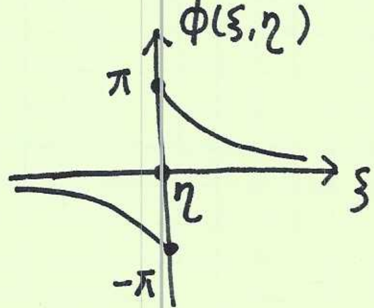
Then $\cot \frac{1}{2} \phi(\xi, \eta) = \frac{1}{2}[\xi - \eta]$ with $\begin{cases} \xi = \cot \frac{k(x)}{2} \\ \eta = \cot \frac{1}{2} k(y) \end{cases}$

Let us fix η: $\xi \to \infty . \leftrightarrow \phi = 0$

$\xi \to \eta + 0^+ \quad \phi = \pi$

$\xi \to \eta + 0^- \quad \phi = -\pi$

$\xi \to -\infty \quad \phi = 0$



$\Rightarrow k(x) = 2\pi x + \int_0^1 dy \cot^{-1} \frac{\xi(x) - \eta(y)}{2}$ ← with the convection that $\cot^{-1} x$ change sign at $x = 0$

$$\frac{dk(x)}{dx} = 2\pi + \underbrace{\int_0^1}_{dy} \left[\frac{-1}{1 + (\xi - \eta)^2/4} + 2\pi \delta(\xi - \eta) \right] \frac{1}{2} \frac{d\xi}{dx}$$

↑ due to the discontinuity

The delta-fun part $\int_0^1 dy\, \delta(\xi - \eta) \frac{d\xi}{dx} = \int_0 dy\, \delta(\xi(x) - \eta(y)) \frac{d\xi}{dx}$

when $\xi(x) = \eta(y)$ ← please notice that ξ and η are same function

$\Rightarrow x = y, \quad \Rightarrow \delta(\xi(x) - \eta(y)) = \frac{\delta(x - y)}{|d\eta/dy|} = \frac{\delta(x-y)}{|d\xi/dx|}$

Since ξ is monotonically decreasing func. $\Rightarrow d\xi/dx < 0$

$\Rightarrow \int_0^1 dy\, \delta(\xi(x) - \eta(y)) \frac{d\xi}{dx} = -1$

$$\Rightarrow \frac{dk(x)}{dx} = \pi + \int_0^1 dy \, \frac{-1}{1+(\xi-\eta)^2/4} \, \frac{1}{2} \frac{d\xi}{dx}$$

change $dy = \frac{dy}{d\eta} d\eta$:

$$\left.\begin{array}{l} \text{as } y: 0 \to 1 \\ k: 0 \to 2\pi \\ \eta: +\infty \to -\infty \end{array}\right\} \eta = \cot \frac{k(y)}{2}$$

$$\Rightarrow \frac{dk(x)}{dx} = \pi + \int_{+\infty}^{-\infty} \frac{-2}{4+(\xi-\eta)^2} \frac{d\xi}{dx} \frac{dy}{d\eta} d\eta = \pi + \left[\int_{-\infty}^{+\infty} \frac{2}{4+(\xi-\eta)^2} \frac{dy}{d\eta} d\eta\right] \cdot \frac{d\xi}{dx}$$

Introduce a function $\boxed{\frac{d\xi(x)}{dx} = -\frac{1}{g(\xi)}}$, $\frac{d\eta(y)}{dy} = -\frac{1}{g(y)}$

we have $$\frac{d\xi(x)}{dx} = \frac{d}{dx}\left[\cot \frac{k(x)}{2}\right] = -\frac{1}{2} \frac{1}{\sin^2 \frac{k(x)}{2}} \frac{dk(x)}{dx}$$

$$= -\frac{1}{2}\left[1 + \cot^2 \frac{k(x)}{2}\right] \frac{dk(x)}{dx} = -\frac{1}{2}(1+\xi^2(x)) \frac{dk(x)}{dx}$$

$$\Rightarrow \frac{dk(x)}{dx} = \frac{2}{1+\xi^2} \frac{1}{g(\xi)}$$

$$\Rightarrow \frac{2}{1+\xi^2} \frac{1}{g(\xi)} = \pi + 2\int_{-\infty}^{+\infty} \frac{d\eta}{4+(\xi-\eta)^2} (-) g(\eta) (-) \frac{1}{g(\xi)}$$

$$\Rightarrow \boxed{\frac{2}{1+\xi^2} = \pi g(\xi) + 2\int_{-\infty}^{+\infty} \frac{g(\eta)}{4+(\xi-\eta)^2} d\eta} \leftarrow$$ $g(\xi)$ is rapidity spectra

This equation can be solved by Fourier transform

define $$G(u) = \int_{-\infty}^{+\infty} g(\xi) \, e^{i\xi u} d\xi$$

and $$g(\xi) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} G(u) e^{-i\xi u} du$$

Remember $\int_{-\infty}^{+\infty} \frac{e^{ikx}}{1+x^2}\, dx = \pi e^{-|k|}$.

we perform Fourier transform:

$$\int_{-\infty}^{+\infty} \frac{2}{1+\xi^2} e^{i\xi u}\, d\xi = \pi \int_{-\infty}^{+\infty} g(\xi)\, e^{i\xi u} d\xi + 2\int d\xi \int_{-\infty}^{+\infty} d\eta\, \frac{g(\eta)\, e^{i\xi u}}{4+(\xi-\eta)^2}\, d\eta$$

$$2\pi e^{-|u|} = \pi g G(u) + \int_{-\infty}^{+\infty} d\eta\, g(\eta)\, e^{i\eta u} \int_{-\infty}^{+\infty} d\frac{\xi}{2}\, \frac{e^{i\frac{\xi-\eta}{2}2u}}{1+\left(\frac{\xi-\eta}{2}\right)^2}$$

$$2\pi e^{-|u|} = \pi G(u) + \int_{-\infty}^{+\infty} d\eta\, g(\eta)\, \pi e^{-2|u|}\, e^{i\eta u} = \pi G(u) + \pi e^{-2|u|} G(u)$$

$$G(u) = \frac{2e^{-|u|}}{1+e^{-2|u|}} = \frac{2}{e^{|u|}+e^{-|u|}} = \boxed{\frac{1}{\cosh u} = G(u)}$$

$$\Rightarrow \frac{E}{NJ} = \frac{1}{2}\int_0^1 dx\,[\cos k(x) - 1] = -\int_0^1 dx\, \sin^2\frac{k(x)}{2} = -\int_{+\infty}^{-\infty} d\xi\, \frac{dx}{d\xi}\, \frac{1}{1+\cot^2\frac{k(x)}{2}}$$

$$= \int_{-\infty}^{+\infty} d\xi\, (-)\, \frac{g(\xi)}{1+\xi^2} = (-)\int_{-\infty}^{+\infty} d\xi \int_{-\infty}^{+\infty} \frac{du}{2\pi}\, \frac{e^{-i\xi u}}{1+\xi^2}\, G(u)$$

$$= -\int_{-\infty}^{+\infty} du\, \frac{\frac{1}{2}e^{-|u|}}{\cosh u} = -\int_0^{+\infty} du\, \frac{e^{-u}}{\cosh u} = -\ln 2$$

$$-\int_1^0 de^{-u} \Big/ \frac{e^{u}+e^{-u}}{2} = 2\int_0^1 \frac{dx}{x+x^{-1}} = \int_0^1 \frac{2x\,dx}{x^2+1} = \ln(x^2+1)\Big|_0^1 = \ln 2$$

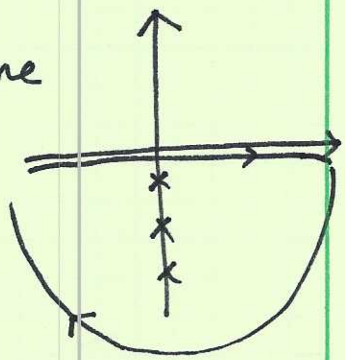
$\Rightarrow$ ground state energy $\boxed{\frac{E}{NJ} = -\ln 2}$.

A few other quantitizes.

① Rapidity (ξ) spectra $\quad g(\xi) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} G(u)\, e^{-i\xi u}\, du = \frac{1}{2\pi}\int_{-\infty}^{+\infty} du \frac{e^{-i\xi u}}{\cosh u}$

The poles of $\frac{1}{\cosh u}$ is at $u_n = i(n+\tfrac{1}{2})\pi$.

If $\xi > 0$, we chose counter in the lower half plane



$$\mathrm{Res}\, \frac{1}{\cosh u}\Big|_{u=u_n} = \frac{1}{\mathrm{sh}\, u}\Big|_{u=u_n} = \frac{1}{-i \sin(n+\frac{1}{2})\pi}$$

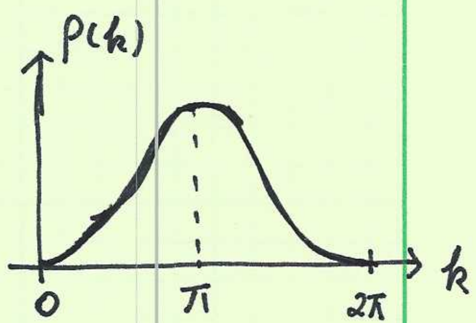
$$= (i)(-)^n$$

$$\Rightarrow g(\xi) = \sum_{n=0}^{+\infty} (-i)(i)(-)^n\, e^{-i\xi\, i(n+1/2)\pi} = \sum_{n=0}^{\infty} (-)^n\, e^{\xi(n+1/2)\pi}$$

$$= \frac{e^{\xi/2\pi}}{1+e^{\xi\pi}} = \frac{1}{e^{-\xi/2\pi}+e^{\xi/2\pi}} = \boxed{\frac{1}{2\cosh\frac{\pi\xi}{2}} = g(\xi)}$$

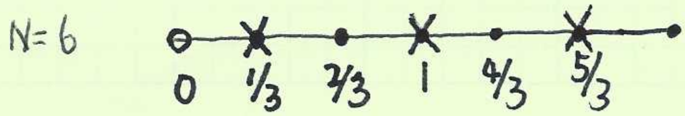
then $\quad \frac{dk(x)}{dx} = \frac{2}{1+\xi^2}\,\frac{1}{g(\xi)} = \frac{2}{1+\cot^2\frac{k}{2}}\, 2\cosh\frac{\pi\xi}{2} = 4\sin^2\frac{k}{2}\cosh\left(\frac{\pi}{2}\cot\frac{k}{2}\right)$

$$\Rightarrow \boxed{\frac{dx}{dk} = \frac{1}{4\sin^2\frac{k}{2}\cosh\left(\frac{\pi}{2}\cot\frac{k}{2}\right)} = \rho(k)}$$



ground state momentum

total momentum 2π



total momentum 3π.

Now use another parameterization of Bethe Quantum # to recalculate the ground state energy.

$$N k_j = 2\pi m_j + \sum_{l \neq j} \phi_{jl}, \qquad m_j \in (0, 1, 2, \cdots N-1) \bmod N$$

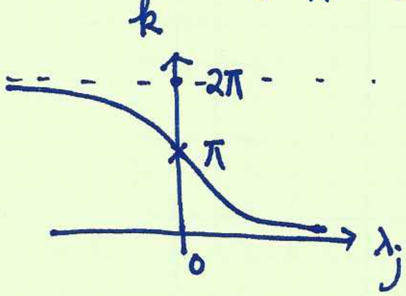
Now express k_j and ϕ_{jl} using rapidities defined $\lambda_j = \frac{1}{2}\cot\frac{k_j}{2}$

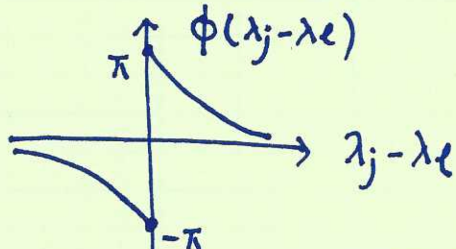
$$2\lambda_j = \cot\frac{k_j}{2} \Rightarrow \frac{k_j}{2} = \frac{\pi}{2} - \tan^{-1} 2\lambda_j \quad \text{or} \quad \boxed{k_j = \pi - 2\tan^{-1} 2\lambda_j}$$

if take $\tan^{-1} 2\lambda_j \in (-\frac{\pi}{2}, \frac{\pi}{2}) \Rightarrow k_j \in [0, 2\pi]$.

$$2\cot\frac{1}{2}\phi_{jl} = \cot\frac{k_j}{2} - \cot\frac{k_l}{2} \Rightarrow \cot\frac{1}{2}\phi_{jl} = \lambda_j - \lambda_l$$

$$\Rightarrow \phi_{jl} = \begin{cases} \pi - 2\tan^{-1}(\lambda_j - \lambda_l) & \text{for } \mathrm{Re}(\lambda_j - \lambda_l) > 0 \\ -\pi - 2\tan^{-1}(\lambda_j - \lambda_l) & \text{for } \mathrm{Re}(\lambda_j - \lambda_l) < 0 \end{cases}$$





$$\boxed{\phi(\lambda_j - \lambda_l) = \pi\,\mathrm{sgn}(\lambda_j - \lambda_l) - 2\tan^{-1}(\lambda_j - \lambda_l)}$$

plug in Bethe equation

$$N[\pi - 2\tan^{-1} 2\lambda_j] = 2\pi m_j + \sum_{l \neq j} \pi\,\mathrm{sgn}(\lambda_j - \lambda_l) - 2\tan^{-1}(\lambda_j - \lambda_l)$$

$$\boxed{\begin{array}{l} \tan^{-1} 2\lambda_j = \frac{\pi}{N} I_j + \frac{1}{N}\sum_{l \neq j} \tan^{-1}(\lambda_j - \lambda_l) \\ \text{where } I_j = \frac{N}{2} - m_j - \frac{1}{2}\sum_{l \neq j} \mathrm{sgn}(\lambda_j - \lambda_l), \leftarrow I_j \text{ takes integers or half integers} \end{array}}$$

mapping $m_j = \{1, 3, 5, \cdots N-1\}$ to the set of I_j

λ is decresing function of k $\Rightarrow \begin{cases} \lambda_1 > \lambda_2 > \lambda_3 > \cdots \\ \Leftrightarrow k_1 < k_2 < k_3 < \cdots \end{cases}$

$$\sum_{l \neq j} \mathrm{sgn}\left(\frac{\lambda_j - \lambda_l}{2}\right) = -(j-1) + \left(\frac{N}{2} - j\right) = \frac{N}{2} - 2j + 1$$

$$\Rightarrow I_j = \frac{N}{2} - m_j - \left[\frac{N}{4} - j + \tfrac{1}{2}\right] = \frac{N}{4} - m_j + j - \tfrac{1}{2} \Bigg\} \Rightarrow \boxed{I_j = \frac{N}{4} - j + \tfrac{1}{2}}$$

plug in $m_j = 2j - 1$

$\Rightarrow I_j = \left\{\frac{N}{4} - \tfrac{1}{2}, \frac{N}{4} - \tfrac{3}{2}, \cdots, -\frac{N}{4} + \tfrac{1}{2}\right\}$, we often reverse the sequence $\Rightarrow \boxed{I_j = \left\{-\frac{N}{4} + \tfrac{1}{2}, -\frac{N}{4} + \tfrac{3}{2}, \cdots, \frac{N}{4} - \tfrac{3}{2}, \frac{N}{4} - \tfrac{1}{2}\right\}}$

Let $x = \frac{I_j}{N} \in [-\tfrac{1}{4}, \tfrac{1}{4}]$. $\Rightarrow \frac{1}{N}\sum_j = \int_{-1/4}^{1/4} dx$

$$\boxed{\tan^{-1} 2\lambda(x) = \pi x + \int_{-1/4}^{1/4} dy \, \tan^{-1}(\lambda(x) - \lambda(y))}$$

define the spectra function of rapidities $\rho(\lambda) = \frac{dx}{d\lambda}$

After reversing the direction of x; λ increases $\rightarrow k$ decrese $\rightarrow x$ increases

or x increas $\rightarrow k$ decreses $\rightarrow \lambda$ increases

$\Rightarrow \rho(\lambda) > 0$. Take derivative

$$\frac{2}{1+4\lambda^2} = \pi \rho(\lambda) + \int_{-1/4}^{1/4} dy \frac{1}{1+(\lambda - \lambda(y))^2}$$

$$= \pi \rho(\lambda) + \int_{-\infty}^{+\infty} d\lambda(y) \frac{dy}{d\lambda(y)} \frac{1}{1+(\lambda - \lambda(y))^2}$$

$$\frac{2}{1+4\lambda^2} = \pi\rho(\lambda) + \int_{-\infty}^{+\infty} d\mu \, \frac{\rho(\mu)}{1+(\lambda-\mu)^2}$$

Again use Fourier transf : $\displaystyle\int_{-\infty}^{+\infty} \frac{d\lambda}{2\pi} \frac{e^{i\omega\lambda}}{a^2+\lambda^2} = \frac{e^{-a|\omega|}}{2a} \quad (a>0)$

define $\boxed{\tilde{\rho}(\omega) = \displaystyle\int_{-\infty}^{+\infty} \frac{d\lambda}{2\pi} e^{i\omega\lambda} \rho(\lambda) \quad \text{and} \quad \rho(\lambda) = \int_{-\infty}^{+\infty} d\omega \; e^{-i\omega\lambda} \tilde{\rho}(\omega)}$

$$\int_{-\infty}^{+\infty} \frac{d\lambda}{2\pi} \frac{2}{1+4\lambda^2} e^{i\omega\lambda} = \pi \int_{-\infty}^{+\infty} \frac{d\lambda}{2\pi} \rho(\lambda) e^{i\omega\lambda} + 2\pi \int_{-\infty}^{+\infty} \frac{d\lambda}{2\pi} \int_{-\infty}^{+\infty} \frac{d\mu}{2\pi} \frac{e^{i\omega(\lambda-\mu)} e^{i\omega\mu} \rho(\mu)}{1+(\lambda-\mu)^2}$$

$$\frac{1}{2} e^{-\frac{1}{2}|\omega|} = \pi\tilde{\rho}(\omega) + 2\pi\tilde{\rho}(\omega) \frac{e^{-|\omega|}}{2} \Rightarrow \tilde{\rho}(\omega) = \frac{1}{2\pi} \frac{e^{-\frac{1}{2}|\omega|}}{1+e^{-|\omega|}}$$

$$\Rightarrow \boxed{\tilde{\rho}(\omega) = \frac{1}{4\pi} \frac{1}{\cosh\frac{\omega}{2}}}$$

$$\rightarrow \rho(\lambda) = \frac{dx}{d\lambda} = \frac{1}{4\pi} \int_{-\infty}^{+\infty} d\omega \, e^{-i\omega\lambda} \frac{1}{\cosh\frac{\omega}{2}} = \frac{1}{2\cosh\pi\lambda}$$

$$\frac{dx}{dk} = \frac{dx}{d\lambda}\frac{d\lambda}{dk} = \rho(\lambda)\, \tfrac{1}{4} \frac{1}{\sin^2 k/2} = \frac{1}{8\sin^2 k/2 \, \cosh(\frac{\pi}{2}\cot k/2)}$$

There's a factor of 2-difference from we obtained before. This is because the normalization $x \in [-1/4, 1/4]$ here.

$$\frac{E}{N} = -\frac{J}{2N} \sum_{j=1}^{N/2} \frac{1}{\lambda_j^2 + 1/4} = -\frac{J}{2} \int_{-1/4}^{1/4} dx \frac{1}{\lambda^2(x)+1/4} = -J/2 \int_{-\infty}^{+\infty} d\lambda \frac{dx}{d\lambda} \frac{1}{\lambda^2+1/4}$$

$\lambda_j \nearrow \cos k \qquad \frac{1}{N}\sum_j \rightarrow \int_{-1/4}^{1/4} dx$

$$= -\frac{J}{2} \int_{-\infty}^{+\infty} d\lambda \frac{\rho(\lambda)}{\lambda^2+1/4} = -J/2 \int_{-\infty}^{+\infty} d\lambda \frac{\frac{\sec \pi\lambda}{2}}{\lambda^2+1/4} = -J/2 \int_{-\infty}^{+\infty} dy \frac{\sec\frac{\pi}{2}y}{y^2+1} = -J\ln 2$$

Two-spinon excitation

Ground state Bethe quantum #s $\{I_i\} = \{-N/4 + 1/2, -N/4 + 3/2, \cdots, N/4 - 1/2\}$

The corresponding pattern for m_j : $\{1, 3, 5, 7, \cdots N-1\}$. $\leftarrow \frac{N}{2}$ particles

For the sector of $S_z = 1$, there're only $N/2 - 1$ particles. The ground state m's pattern : $\boxed{m_j = \{2, 4, \cdots N-2\},}$ which is more close to center, or $m_j = 2j$ for $j = 1, \cdots \frac{N}{2} - 1$.

Again for $k_1 < k_2 < \cdots$, we have $\lambda_1 > \lambda_2 > \cdots$. we have

$$\sum_{l \neq j} \operatorname{sgn}\left(\frac{\lambda_j - \lambda_l}{2}\right) = -(j-1) + \left(\frac{N}{2} - S^z - j\right) = \frac{N}{2} - 2j - S^z + 1$$

$$I_j = \frac{N}{2} - m_j - \frac{1}{2}\sum_{l \neq j} \operatorname{sgn}\frac{\lambda_j - \lambda_l}{2} = \frac{N}{4} - m_j + j + \frac{S_z}{2} - \frac{1}{2} \rightarrow \boxed{I_j = \frac{N}{4} - (m_j - j)}$$

for $S_z = 1$

For the ground state with $S_z = 1$, the $\{I_i\} = \frac{N}{4} - j$ for $j = 1, \cdots \frac{N}{2} - 1$

$$\Rightarrow \boxed{\{I_i\} = \left\{-\frac{N}{4} + 1, \cdots \frac{N}{4} - 1\right\}.}$$

How about the excitation spectra? We have $1 \leq m_j \leq N-1$. (If $m_1 = 0$, usually it means it comes from S^- applying on the ferro-magnetic state.

$$\Rightarrow I_1 = \frac{N}{4} + 1 - m_j \leq \frac{N}{4}, \quad I_{\frac{N}{2}-1} = \frac{N}{4} - m_j + \left(\frac{N}{2} - 1\right) \geq \frac{3}{4}N - 1 - (N-1) = -\frac{N}{4}.$$

So for the range of $1 \leq i \leq \frac{N}{2} - 1$, $\Rightarrow -\frac{N}{4} \leq I_i \leq \frac{N}{4}$.

There're $\frac{N}{2} + 1$ possible values of I_i, and we need to take $\frac{N}{2} - 1$ among them, or, there are two holes to specify an excited state in this sector.

Suppose we have two jumps for m_j at the sth and tth magnon

$$m_j = \{1, 3, \cdots, 2s-3, \underbrace{\ }_{\text{1st jump}} 2s, 2s+2, \cdots 2t-2, \underbrace{\ }_{\text{2nd jump}} 2t+1, \cdots N-1\}$$

$$\Rightarrow m_j = \begin{cases} 2j-1 & \text{for } i \le s-1 \\ 2j & \text{for } s \le i \le t-1 \\ 2j+1 & \text{for } i \ge t \end{cases} \quad \text{then } I_i = \begin{cases} \frac{N}{4} - i + 1, & i \le s-1 \\ \frac{N}{4} - i & s \le i \le t-1 \\ \frac{N}{4} - i - 1 & i \ge t \end{cases}$$

or $I_j = \{\frac{N}{4}, \cdots, \frac{N}{4} - s + 2, \} \frac{N}{4} - s, \cdots \cdot \frac{N}{4} - t + 1 \} \frac{N}{4} - t - 1, \cdots - \frac{N}{4}\}$.

the corresponding configuration

* Now let us solve Bethe ansatz Eq

$$\boxed{\begin{aligned} & N k_j = 2\pi m_j + \sum_{l \ne j} \phi_{jl} \\ & 2\lambda_j = \cot \frac{k_j}{2} \Rightarrow k_j = \pi - 2\tan^{-1} 2\lambda_j \\ & 2\cot \frac{1}{2} \phi_{jl} = \cot \frac{k_j}{2} - \cot \frac{k_l}{2} \Rightarrow \phi(\lambda_j - \lambda_l) = \pi \,\mathrm{sgn}(\lambda_j - \lambda_l) - 2\tan^{-1}(\lambda_j - \lambda_l) \end{aligned}}$$

$$\rightarrow \tan^{-1} 2\lambda_j = \frac{\pi}{N} I_j + \frac{1}{N} \sum_{l \ne j} \tan^{-1}(\lambda_j - \lambda_l)$$

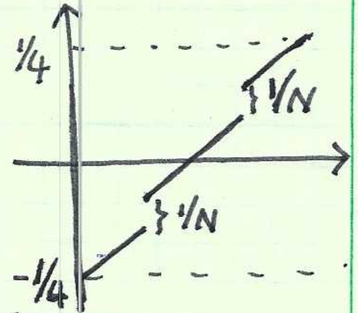
and we revese the sequence of I_j such that I_j is increasing

set $x = \frac{i}{N}$ and $y = \frac{I_i}{N}$, since $I_i = -\frac{N}{4} - 1 + i + \Theta(i - S_1) + \Theta(i - S_2)$

(S_1, S_2 are related with s and t, Here we reorder them for simplicity)

$$\Rightarrow y_{(x)} = -\frac{1}{4} - \frac{1}{N} + x + \frac{1}{N}\Theta(x - x_1) + \frac{1}{N}\Theta(x - x_2), \quad \text{where } x_1 = \frac{S_1}{N}, \ x_2 = S_2/N.$$

$$\tan^{-1} 2\lambda(x) = \pi y(x) + \int_{-1/4}^{1/4} dx' \tan^{-1}\left(\lambda(x) - \lambda(x')\right)$$



$$\frac{2}{1+4\lambda^2} = \pi \frac{dy}{dx}\frac{dx}{d\lambda} + \int_{-1/4}^{1/4} dx' \frac{1}{1+(\lambda-\lambda(x'))^2}$$

$$= \pi \frac{dx}{d\lambda}\left(1 + \frac{1}{N}\delta(x-x_1) + \frac{1}{N}\delta(x-x_2)\right] + \int_{-\infty}^{+\infty} d\lambda' \frac{1}{1+(\lambda-\lambda')^2} \frac{dx'}{d\lambda}$$

$$= \pi \frac{dx}{d\lambda} + \frac{\pi}{N}\delta(\lambda-\lambda_1(x_1)) + \frac{\pi}{N}\delta(\lambda-\lambda_2(x_2)) + \int_{-\infty}^{+\infty} d\lambda' \frac{1}{1+(\lambda-\lambda')^2}\frac{dx'}{d\lambda}$$

$$\frac{2}{1+4\lambda^2} - \frac{\pi}{N}\delta(\lambda-\lambda_1(x_1)) - \frac{\pi}{N}\delta(\lambda-\lambda_2(x_2)) = \pi\frac{dx}{d\lambda} + \int_{-\infty}^{+\infty} d\lambda' \frac{1}{1+(\lambda-\lambda')^2}\frac{dx'}{d\lambda}$$

again $p(\lambda) = \frac{dx}{d\lambda}$, $\tilde{p}(\omega) = \int \frac{d\lambda}{2\pi} e^{i\omega\lambda} p(\lambda)$.

$$\frac{1}{2} e^{-\frac{1}{2}|\omega|} - \frac{\pi}{N}\left[e^{i\omega\lambda_1} + e^{i\omega\lambda_2}\right] = \pi \tilde{p}(\omega)\left[1 + e^{-|\omega|}\right]$$

$$\Rightarrow \tilde{p}(\omega) = \tilde{p}^{(0)}(\omega) - \frac{1}{N}\frac{e^{i\omega\lambda_1} + e^{i\omega\lambda_2}}{1+e^{-|\omega|}}, \text{ where } \tilde{p}^{(0)}(\omega) = \frac{1}{4\pi}\frac{1}{\cosh \omega/2}$$

Now calculate the excitatin energy

$$E = -\frac{J}{2}\sum_{j=1}^{N/2-1} \frac{1}{\lambda_j^2 + 1/4} = -\frac{JN}{2}\int_{-1/4}^{1/4} dx \frac{1}{\lambda^2(x)+1/4} = -\frac{JN}{2}\int_{-\infty}^{+\infty} d\lambda \frac{dx}{d\lambda}\frac{1}{\lambda^2+1/4}$$

$$\Delta E = -\frac{JN}{2}\int_{-\infty}^{+\infty} d\lambda \left(-\frac{1}{N}\right)\int_{-\infty}^{+\infty} d\omega\, e^{-i\omega\lambda} \frac{e^{i\omega\lambda_1}+e^{i\omega\lambda_2}}{1+e^{-|\omega|}}\frac{1}{\lambda^2+1/4}$$

$$\Delta E = \frac{J}{2}\int_{-\infty}^{+\infty} d\omega \frac{e^{i\omega\lambda_1}+e^{i\omega\lambda_2}}{1+e^{-|\omega|}} \boxed{\int_{-\infty}^{+\infty} d\lambda \frac{e^{-i\omega\lambda}}{\lambda^2+1/4}} \leftarrow e^{-\frac{1}{2}|\omega|}$$

$$\Delta E = \frac{J}{2}\int_{-\infty}^{+\infty} d\omega \; \frac{e^{i\omega\lambda_1} + e^{i\omega\lambda_2}}{2\cosh\frac{\omega}{2}} = \frac{J}{2}\int_{-\infty}^{+\infty} d\omega \; \frac{e^{i2\omega\lambda_1} + e^{i2\omega\lambda_2}}{\cosh\omega}$$

use $$\frac{1}{2\pi}\int_{-\infty}^{+\infty} du \; \frac{e^{-i\xi u}}{\cosh u} = \frac{1}{2\operatorname{ch}\frac{\pi\xi}{2}}$$

$$\Rightarrow \boxed{\Delta E = \frac{J}{2}\pi\left[\frac{1}{\cosh\pi\lambda_1} + \frac{1}{\cosh\pi\lambda_2}\right]}$$

It's interesting that excitations are summed $\rightarrow$ just two holes.

But we still need to determine the values of λ_1 and λ_2 at the discontinous point. At $\lambda \to -\infty$, $k \to$ largest, $x \to \frac{\frac{N}{2}-1}{N} \to 1/2$.

$$x(\lambda_1) - x(-\infty) = \int_{-\infty}^{\lambda_1} \frac{dx}{d\lambda}\, d\lambda\,,$$ we can use the zeroth order value

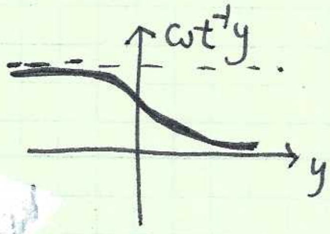
$$\rho^0(\lambda) = \frac{1}{2\cosh\pi\lambda} \qquad \Rightarrow x(\lambda_1) = \frac{1}{2} - \int_{-\infty}^{\lambda_1} \frac{d\lambda}{2\cosh\pi\lambda}$$

define $y_0 = \sinh\pi\lambda \Rightarrow dy = \pi\cosh\pi\lambda\, d\lambda$

$$\Rightarrow x(\lambda_1) = 1/2 - \int_{-\infty}^{y_1} \frac{dy}{2\pi(1+y^2)}$$ where $y_1 = \sinh\pi\lambda_1$.

$$= 1/2 + \frac{1}{2\pi}\cot^{-1} y \Big|_{-\infty}^{y_1} = \frac{1}{2\pi}\cot^{-1} y_1 + \left[1/2 - \frac{1}{2\pi}\cot^{-1} y\Big|_{y\to-\infty}\right]$$

We use $\cot^{-1} y\big|_{y\to-\infty} \longrightarrow \pi$

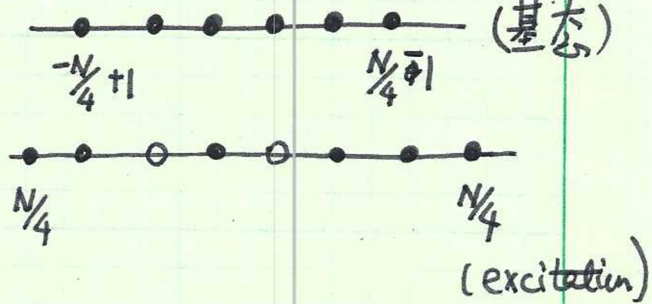


$$\Rightarrow x(\lambda_1) = \frac{1}{2\pi}\cot^{-1}(\sinh\pi\lambda_1)$$

What's momentum of the system

acerding to the relation $I_j = \frac{N}{4} - (m_j - j)$

$$\sum_{j=1}^{\frac{N}{2}-1} k_j = \frac{2\pi}{N} \sum_{j=1}^{N/2-1} m_j = \frac{2\pi}{N} \sum_{j=1}^{N/2-1} (N/4 + j) - \frac{2\pi}{N} \sum_{j=1}^{N/2-1} I_j$$



$$\sum_{j=1}^{N/2-1} \frac{N}{4} + j = \frac{N}{4}\left(\frac{N}{2}-1\right) + \left(\frac{N}{2}-1\right)\left(\frac{N}{2}\right)\frac{1}{2}$$

$$= \frac{N}{2}\left(\frac{N}{2}-1\right)$$

$$\Rightarrow \sum_{j=1}^{\frac{N}{2}-1} k_j = \pi\left(\frac{N}{2}-1\right) - \frac{2\pi}{N} \sum_{j=1}^{\frac{N}{2}-1} I_j$$

$$\Rightarrow \boxed{\Delta K = \frac{2\pi}{N} s_1 + \frac{2\pi}{N} s_2}$$

$$\chi(\lambda_1) = \frac{s_1}{N} = \frac{q_1}{2\pi} \Rightarrow \frac{q_1}{2\pi} = \frac{1}{2\pi} \cot^{-1}(\sinh \pi \lambda_1)$$

$$\cot q_1 = \sinh \pi \lambda_1 \Rightarrow \frac{1}{\cosh \pi \lambda_1} = \frac{1}{\sqrt{1 + \sinh^2 \pi \lambda_1}}$$

$$= \frac{1}{\sqrt{1 + \cot^2 q_1}} = |\sin q_1|$$

similarly calculation can be perfirmed

at the 2nd discontineus point

$$\begin{cases} \chi(\lambda_2) = \frac{s_2}{N} = \frac{q_2}{2\pi} \\ \chi(\lambda_2) = \frac{1}{2\pi} \cot^{-1}(\sinh \pi \lambda_2) \end{cases} \Rightarrow \quad \cot q_2 = |\sin q_2|$$

$$\Rightarrow \boxed{\Delta K = q_1 + q_2, \text{ with } \Delta E = \frac{J\pi}{2}\left[|\sin q_1| + |\sin q_2|\right].}$$

$$\boxed{\text{with } q_1 = \frac{2\pi}{N} s_1, \; q_2 = \frac{2\pi}{N} s_2}$$