3 : 3a) the zero of dielectric function $\varepsilon(q.\omega)$ determines the excitation (plasmon spectrum)

$$\varepsilon(q,\omega) = 1 + \frac{4\pi e^2}{q^2} N(0) \int \frac{d\Omega}{4\pi} \frac{-\cos\theta}{s - \cos\theta + i\eta} \qquad \left(s = \frac{\omega}{v_F q} \right)$$

at $q \to 0$.

at $s \gg 1$
$$\frac{-\cos\theta}{s-\cos\theta} = \frac{-\cos\theta/s}{1-\cos\theta/s} = -\frac{\cos\theta}{s}\left(1 + \frac{\cos\theta}{s} + \left(\frac{\cos\theta}{s}\right)^2 + \left(\frac{\cos\theta}{s}\right)^3\right)$$

$$\Rightarrow \varepsilon(q.\omega) = 1 + \frac{4\pi e^2}{q^2} N(0) \left(\frac{-1}{3s^2} - \frac{1}{5s^4} \right)$$

keep to $1/s^2$ order, we have
$$1 = \frac{4\pi e^2 N(0)\, v_F^2}{3\ \omega^2} \Rightarrow \omega^2 = \omega_p^2$$

keep to $1/s^4$ order

$$\varepsilon(q.\omega) = 1 - \left(\frac{\omega_p^2}{\omega^2} + \frac{3}{5}\frac{\omega_p^2}{\omega^4}(v_F q)^2 \right) = 0$$

$$\frac{\omega_p^2}{\omega^2} = 1 - \frac{3}{5}\left(\frac{\omega_p}{\omega}\right)^2\left(\frac{v_F q}{\omega}\right)^2 \Rightarrow \frac{\omega^2}{\omega_p^2} \approx 1 + \frac{3}{5}\left(\frac{\omega_p}{\omega}\right)^2 \left(\frac{v_F q}{\omega}\right)^2$$

$$\approx 1 + \frac{3}{5}\left(\frac{v_F q}{\omega_p}\right)^2$$

3b)
$$\frac{\partial n}{\partial t} + \nabla\cdot(n\vec{v}) = 0$$

$$\frac{\partial^2}{\partial t^2} n + \nabla\cdot\frac{\partial}{\partial t}(n\vec{v}) = 0\,,$$

from $$m\frac{\partial}{\partial t}(n\vec{v}) + m\vec{v}\cdot\nabla(n\vec{v}) = -ne\vec{E}$$

$$\nabla\cdot\frac{\partial}{\partial t}(n\vec{v}) + \nabla\left(\vec{v}\cdot\nabla(n\vec{v})\right) = -\frac{\nabla}{m}(ne\vec{E})$$

$$\Rightarrow \frac{\partial^2}{\partial t^2} n = \left(\nabla(\vec{v}\cdot\nabla(n\vec{v})\right) + \frac{\nabla}{m}(ne\vec{E})$$

$\nabla(\vec{v}\cdot\nabla(n\vec{v}))$ gives corrections at the order of $k^2 \rightarrow$ neglect eel.

$$\frac{\partial^2}{\partial t^2}(n_0 + \delta n) = \frac{\nabla}{m}(e(n+\delta n)\vec{E}) = \frac{e}{m}\nabla(n+\delta n)\vec{E} + e(n+\delta n)\frac{\nabla\vec{E}}{m}$$

$$\nabla\vec{E} = -4\pi e\,\delta n \quad \Rightarrow \quad \vec{E} \propto \delta n,$$ keep to linear order

$$\frac{\partial^2}{\partial t^2}\delta n = -\frac{4\pi e^2}{m} n_0\,\delta n \quad \Rightarrow \quad \omega_p^2 = \frac{4\pi n_0 e^2}{m}$$

4): (a) $$\delta \mathcal{E}_{HF}(k) = -\frac{1}{V}\sum_q V_q\, n_{k+q} = -\int \frac{d^3q}{(2\pi)^3}\frac{4\pi e^2}{q^2+k_{TF}^2}\, n_{k+q}$$

$$k_{TF} = 4\pi e^2 \frac{\partial n}{\partial \mu}.$$

$$\left(\frac{k_{TF}}{k_f}\right)^2 = \frac{4}{\pi}\frac{1}{k_f a_0}, \qquad \text{define } \frac{4}{3}\pi r_s^3 = 1/n \Rightarrow k_f = \left(\frac{9}{4}\pi\right)^{1/3} r_s^{-1}$$

$$\Rightarrow \left(\frac{k_{TF}}{k_f}\right)^2 = \frac{4}{\pi}\left(\frac{4}{9\pi}\right)^{1/3}\frac{r_s}{a_0} \approx 0.7\,\frac{r_s}{a_0}$$

in the dilute limit $q \propto k_F \propto \frac{1}{r_s}$, $k_{TF} \propto r_s^{-1/2} \Rightarrow k_{TF} \gg k_F$

we can neglect $q^2 \Rightarrow q^2 + k_{TF}^2 \approx k_{TF}^2$

$$\Rightarrow \delta \mathcal{E}_{HF}(k) = -\frac{1}{V}\frac{4\pi e^2}{k_{TF}^2}\sum_q n_{k+q} = -\frac{n}{2}\frac{4\pi e^2}{k_{TF}^2}$$

↑ only sum over particles with the same spin.

b) with a finite polarization

$$\delta \mathcal{E}_{HF\uparrow}(k) = -\frac{1}{V}\frac{4\pi e^2}{k_{TF}^2}\sum_q n_{k+q\uparrow} = -\frac{n}{2}(1+p)\frac{4\pi e^2}{k_{TF}^2(p)}$$

$$\delta \mathcal{E}_{HF\downarrow}(k) = -\frac{n}{2}(1-p)\frac{4\pi e^2}{k_{TF}^2(p)}$$

$k_{TF}^2(p)$ is the the T-F screening wave vector with polarization p

$$k_{TF} = 4\pi e^2\left[\left(\frac{\partial n}{\partial \mu}\right)_\uparrow + \left(\frac{\partial n}{\partial \mu}\right)_\downarrow\right] \Rightarrow \frac{k_{TF}(p)}{k_{TF}} = \frac{\left(\frac{\partial n}{\partial \mu}\right)_\uparrow + \left(\frac{\partial n}{\partial \mu}\right)_\downarrow}{\left(\frac{\partial n}{\partial \mu}\right)(\text{at } p=0)}$$

we know $\frac{\partial n}{\partial \mu} \propto \frac{4\pi k_F^2}{v_F} \propto k_F$ in 3D $\Rightarrow$

$$k_{TF}(p)/k_{TF} = \left[(1+p)^{1/3} + (1-p)^{1/3}\right]/2 = 1 - \tfrac{1}{9} p^2$$

$$\Rightarrow \delta \mathcal{E}_{HF\uparrow}(k) = -\frac{n}{2} \frac{1+p}{1-\frac{1}{9}p^2} \frac{4\pi e^2}{k_{TF}^2(p=0)} \quad , \delta \mathcal{E}_{HF\downarrow}(k) = -\frac{n}{2} \frac{1-p}{1-\frac{1}{9}p^2} \frac{4\pi e^2}{k_{TF}^2(p=0)}$$

at small p.

The total HF energy

$$E_{ex\uparrow} = \frac{N_\uparrow}{2} \delta \mathcal{E}_{HF}(k_f) = -\frac{V}{2} n_\uparrow^2 \frac{4\pi e^2}{k_{TF}^2(p)} \quad , \quad E_{ex\downarrow} = -\frac{V}{2} n_\downarrow^2 \frac{4\pi e^2}{k_{TF}^2(p=0)}$$

$$\Rightarrow \frac{E_{ex}}{V} = -\frac{1}{2} \frac{1}{4} n^2 \left[(1+p)^2 + (1-p)^2\right] (1 - \tfrac{1}{9}p^2)^{-1} \frac{4\pi e^2}{k_{TF}^2(p=0)}$$

$$= -\frac{n^2}{4} (1 + \frac{10}{9} p^2) \cdot \frac{4\pi e^2}{k_{TF}^2(p=0)}$$

The kinetic energy with Polarization p should be.

$$E_{K\uparrow} = \frac{3}{5} N_\uparrow \mathcal{E}_{f\uparrow}^0 = \frac{3}{5} \frac{1+p}{2} N_0 \mathcal{E}_f^0 (1+p)^{\frac{2}{3}}$$

$$E_{K\downarrow} = \frac{3}{5} \frac{1-p}{2} N \mathcal{E}_f^0 (1-p)^{2/3}$$

$$\Rightarrow E_K = \frac{3}{5} N \mathcal{E}_f^0 \left[(1+p)^{5/3} + (1-p)^{5/3}\right] = \frac{3}{5} N \mathcal{E}_f^0 (1 + \tfrac{5}{9} p^2)$$

$$\Rightarrow \frac{E_{tot}}{V} = \frac{3}{5} n \mathcal{E}_f^0 (1 + \frac{5}{9} p^2) - \frac{n^2}{4} (1 + \frac{10}{9} p^2) \frac{4\pi e^2}{k_{TF}^2(p=0)}$$

d) $\frac{dE}{V} = B\frac{dM}{V} = \mu_B B\, n\, dp$

$$\mu_B B\cdot n = \frac{\partial E/V}{\partial p} = \frac{2}{3} n \varepsilon_f^0 p - \frac{5}{9} n^2 p \cdot \frac{4\pi e^2}{k_{TF}^2(p=0)}$$

$$\chi_{HF} = \frac{M}{B} = \frac{n\mu_B p}{B} = \frac{n^2\mu_B^2 p}{\mu_B B\cdot n} = \frac{n^2\mu_B^2 p}{\frac{2}{3} n\varepsilon_f^0 p - \frac{5}{9} n^2 p\frac{4\pi e^2}{k_{TF}^2(p=0)}}$$

$$= \frac{n\mu_B^2}{\frac{2}{3}\varepsilon_f^0 - \frac{5}{9} n\frac{4\pi e^2}{k_{TF}^2(p=0)}}$$

For free fermion $\Rightarrow \quad \chi_0 = \frac{n\mu_B^2}{\frac{2}{3}\varepsilon_f^0}$

$$\Rightarrow \quad \frac{\chi_{HF}}{\chi} = \frac{\frac{2}{3}\varepsilon_f^0}{\frac{2}{3}\varepsilon_f^0 - \frac{5}{9} n\frac{4\pi e^2}{k_{TF}^2(p=0)}} = \frac{1}{1 - \frac{5}{6}\frac{n}{\varepsilon_f^0}\frac{4\pi e^2}{k_{TF}^2}}$$

$$\frac{n}{\varepsilon_f}\cdot\frac{4\pi e^2}{k_{TF}^2} = \frac{n}{\varepsilon_f^0}\cdot\frac{1}{\frac{\partial n}{\partial \mu}} = \frac{n}{\varepsilon_f^0}\cdot\frac{1}{\frac{3}{2}\frac{n}{\varepsilon_f^0}} = \frac{2}{3}$$

$$\frac{\chi_{HF}}{\chi} = \frac{1}{1 - 5/6\cdot\frac{2}{3}} \simeq \frac{1}{4/9} = 2.25$$

whether ferromagnetism can occur in ~~a~~ electron gas is a subtle issue, and we will neglect it.