# A systematic study of simple representations of SU(N)

We will consider the one column fully anti-symmetric Rep, one row fully symmetric Rep, the adjoint Rep. etc.

## ① A systemetic construction of the Gell-man matrices for SU(N)

We define $\sigma_1$-type $(T^{(1)}_{ab})_{cd} = \frac{1}{2}[\delta_{ac}\delta_{bd} + \delta_{bc}\delta_{ad}]$ $a<b$

$\sigma_2$-type $(T^{(2)}_{ab})_{cd} = -\frac{i}{2}[\delta_{ac}\delta_{bd} - \delta_{bc}\delta_{ad}]$ $a<b$

"a b" denote which matrix, and cd are the row and column indices.

$$T^{(1)}_{ab} = \begin{array}{c|cc} & a & b \\ \hline a & & 1/2 \\ b & 1/2 & \end{array}, \qquad T^{(2)}_{ab} = \begin{array}{c|cc} & a & b \\ \hline a & 0 & -i/2 \\ b & i/2 & 0 \end{array}$$

$T^{(3)}$ - $\sigma_3$ type matrix

$a = 2, \cdots N$

$$(T^{(3)}_a)_{cd} = \begin{cases} \delta_{cd}\left[\frac{1}{2a(a-1)}\right]^{1/2} & c<a \\ -\delta_{cd}\left[\frac{a-1}{2a}\right]^{1/2} & c=a \\ 0 & c>a \end{cases}$$

$$T^{(3)}_2 = \frac{1}{2}\begin{bmatrix} 1 & & \\ & -1 & \\ & & 0 & \\ & & & \ddots \end{bmatrix}, \quad T^{(3)}_3 = \frac{1}{2\sqrt{3}}\begin{bmatrix} 1 & & \\ & 1 & \\ & & -2 & \\ & & & \ddots \end{bmatrix} \quad T^{(4)}_4 = \frac{1}{2\sqrt{6}}\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -3 & \\ & & & & \ddots \end{bmatrix}$$

we often label them in the sequence of

3 $T_1 = T^{(1)}_{12}$ $T_2 = T^{(2)}_{12}$ $T_3 = T^{(3)}_2$ — 12; 2

5 $T_4 = T^{(1)}_{13}$ $T_5 = T^{(2)}_{13}$, $T_6 = T^{(1)}_{23}$, $T_7 = T^{(2)}_{23}$, $T_8 = T^{(3)}_3$ — 13, 23,

7 $T_9 = T^{(1)}_{14}$ $T_{10} = T^{(2)}_{14}$, $T_{11} = T^{(1)}_{24}$, $T_{12} = T^{(2)}_{24}$ $T_{13} = T^{(1)}_{34}$ $T_{14} = T^{(2)}_{34}$ $T_{15} = T^{(3)}_4$

⇖ 14, 24, 34, 4

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This normalization satisfies

$$\boxed{\mathrm{Tr}[T_A T_B] = \frac{1}{2}\delta_{AB}}$$

The above Gellmann matrices are for the fundematal Rep $\square$. The basis is denoted as $|i\rangle = a_i^{+}|\Omega\rangle$, $i=1,2,\cdots N$. It's dimensin is N.

Let's calculate it Casimir: Acording to $C_2(\square) = T_2(\square)\cdot g/m_\square$

$$\mathrm{Tr}[T_A T_B] = T_2(\square)\,\delta_{AB} \Rightarrow T_2(\square) = 1/2.$$

hence $$\boxed{C_2(\square) = \frac{1}{2N}(N^2-1)}$$

② The weight diagram of the fundamental Rep $\square$

$$H_1 = T_2^{(3)},\quad H_2 = T_3^{(3)},\quad \cdots \quad H_{N-1} = T_N^{(3)}$$

state

$$|1\rangle \rightarrow \left(1/2,\ \frac{1}{2\sqrt{3}},\ \cdots \left(\frac{1}{2i(i+1)}\right)^{1/2},\ \cdots\left(\frac{1}{2(N-1)N}\right)^{1/2}\right]$$

$$|2\rangle \rightarrow \left[-1/2,\ \frac{1}{2\sqrt{3}},\ \cdots \left(\frac{1}{2i(i+1)}\right)^{1/2}\cdot \cdot\left(\frac{1}{2(N-1)N}\right)^{1/2}\right]$$

$$|3\rangle \rightarrow \left[0,\ -\frac{1}{\sqrt{3}},\ \cdots \left(\frac{1}{2i(i+1)}\right)^{1/2} \cdots \left(\frac{1}{2(N-1)N}\right)^{1/2}\right]$$

$$\vdots$$

$$|j\rangle \rightarrow \Big[\underbrace{0,\ \cdots 0}_{j-2},\ -\left[\frac{j-1}{2j}\right]^{1/2},\ \cdots \left(\frac{1}{2i(i+1)}\right)^{1/2},\ \cdots \left(\frac{1}{2(N-1)N}\right)^{1/2}$$

$$\vdots$$

$$|N\rangle \rightarrow \Big[\underbrace{0,\ \cdots\cdots\ 0}_{N-1},\ -\left[\frac{N-1}{2N}\right]^{1/2}$$

The square of norms of these weight vectors are $$\boxed{\frac{j-1}{2j} + \sum_{i=j}^{N-1} \frac{1}{2i(i+1)} = \frac{N-1}{2N}}$$

The inner product of two vectur $\vec{i}_1 \cdot \vec{j}_2$ (assume $j_2 > j_1$ )
and $j_2 \geq 3$

$$\vec{j}_1 \cdot \vec{j}_2 = -\left(\frac{1}{2(j_2-1)j_2} \cdot \frac{j_2-1}{2j_2}\right)^{1/2}$$

$$+\sum_{i=j_2}^{N-1} \frac{1}{2i(i+1)} = -\frac{1}{2j_2} + \frac{1}{2}\left[\frac{1}{j_2} - \frac{1}{N}\right] = -\frac{1}{2N}$$

Hence the angle $\cos^{-1} \vec{j}_1\cdot\vec{j}_2 \Big/ j_1\cdot j_1 = -\frac{1}{2N}\Big/\frac{N-1}{2N} = -\frac{1}{N-1}$

For $SU(2) \to 180^\circ$

$SU(3) \to 120^\circ$ …

$SU(4) \to \cos^{-1}(-1/3)$

$\boxed{\text{Hence the weight vector of the SU(N) fundamental Rep forms the N-1 dimensional simplex.}}$

② the one coloumn fully anti-symmetric Reps.

$$r\left\{\begin{array}{|c|}\hline \\ \hline \\ \hline \\ \hline\end{array}\right. \quad r \leq N. \quad \text{dimension} = \binom{N}{r}$$

The basis can be denoted $|\phi_{(i_1 i_2 i_3)\cdots i_r}\rangle = a^\dagger_{i_1} a^\dagger_{i_2} \cdots a^\dagger_{i_r} |\Omega\rangle$

with $i_1 \leq i_2 \cdots \leq i_r$. Here $a^\dagger_i$ are creation operators for fermions.

what's $Tr[I_a^{[1]^r} I_b^{[1]^r}] = T_2[[1]^r]\,\delta_{ab}$ ?

Let's take $I_3 = \frac{1}{2}\left[C_1^\dagger C_1 - C_2^\dagger C_2\right]$ to evaluate $T_2[(1)^r]$.

Let's classify $|\phi_{i_1\cdots i_r}\rangle$ into three categories:

① $i_1 = 1, \quad i_2 = 2,$ then $I_3 = 0$

② $i_1 = 1, \quad i_2 \geq 3$ then $I_3 = 1/2$, there're $\binom{N-2}{r-1}$ states

③ $i_1 = 2$ then $I_3 = -1/2$, there're $\binom{N-2}{r-1}$ states

④ $i_1 = 3$, then $I_3 = 0$

$$\Rightarrow \mathrm{Tr}\left[ I_3^{[1]^r} \cdot I_3^{[1]^r} \right] = \sum_{i_1\cdots i_r} \left( \langle \phi_{i_1\cdots i_r} | \tfrac{1}{2}(c_1^{\dagger} c_1 - c_2^{\dagger} c_2) | \phi_{i_1\cdots i_r} \rangle \right)^2$$

$$= \tfrac{1}{4} \cdot \binom{N-2}{r-1} \cdot 2 = \frac{1}{2} \binom{N-2}{r-1}$$

Hence $\mathrm{Tr}\left[ I_a^{[1]^r} I_b^{[1]^r} \right] = T_2[[1]^r]\, \delta_{ab}$ with $\boxed{T_2[[1]^r] = \frac{1}{2}\binom{N-2}{r-1}}$

$$\Rightarrow C_2[[1]^r] = \frac{g}{m[[1]^r]} \cdot T_2[[1]^r] = \frac{N^2-1}{\binom{N}{r}} \cdot \frac{1}{2} \binom{N-2}{r-1}$$

$$= \frac{N^2-1}{2} \cdot \frac{r!(N-r)!}{N!} \cdot \frac{(N-2)!}{(N-r-1)!\,(r-1)!} = \frac{r(N-r)(N+1)}{2N}$$

hence, the Casmir for  $\}r \leq N$, its Casimir $\dfrac{r(N-r)(N+1)}{2N}$.

③ Now we study the N-D dimensional harmonic oscillater Reps — one row Rep

$$\underbrace{\boxed{\phantom{x}}\boxed{\phantom{x}}\boxed{\cdots}}_{r} \qquad \dim([r]) = \frac{\boxed{N}\boxed{N+1}\boxed{\cdots}}{\boxed{r}\boxed{r-1}\boxed{\cdots}} = \frac{(N+r-1)!}{r!(N-1)!} = \binom{N+r-1}{r}$$

how to calculate Casimir? Let us pick out a basis $|\phi\rangle = \frac{1}{\sqrt{r!}}(a_N^{\dagger})^r|$

Such a state is $T_2, T_3, \cdots T_N$ eigenstate, it's eigenvalues are

$$T_2: 0, \quad T_3: 0, \quad \cdots \quad T_N: -r\left(\frac{N-1}{2N}\right)^{1/2}, \text{ hence } T_2^2 + \cdots T_N^2 = \frac{N-1}{2N}r^2.$$

Then consider $(T_{1N}^{(1)})^2 + (T_{1N}^{(2)})^2 = \frac{1}{2}\{[T_{1N}^{(1)} + i T_{1N}^{(2)}][T_{1N}^{(1)} - i T_{1N}^{(2)}]$

$$+ [T_{1N}^{(1)} - i T_{1N}^{(2)}][T_{1N}^{(1)} + i T_{1N}^{(2)}]\}$$

hence in the subspace 1 and N, the transvers operaters behaves just as SU(2) $\Rightarrow (T_{1N}^{(1)})^2 + (T_{1N}^{(2)})^2 = \frac{r}{2}(\frac{r}{2}+1) - (\frac{r}{2})^2 = \frac{r}{2}$

We can also consider $(T_{2N}^{(1)})^2 + (T_{2N}^{(2)})^2, \cdots (T_{N-1,N}^{(1)})^2 + (T_{N-1,N}^{(2)})^2$

$\Rightarrow$ there're $(N-1)\cdot \frac{r}{2}$

$$\Rightarrow \boxed{C_2[[r]] = \frac{N-1}{2N}r^2 + \frac{r}{2}(N-1) = \frac{N-1}{2}r\left[\frac{r}{N}+1\right]}$$

1° For SU(2) $\quad C_2[r] = \frac{r}{2}\left[\frac{r}{2}+1\right] = s(s+1)$ with $s = \frac{r}{2}$

2° For SU(3) $\quad C_2[r] = r\left[\frac{r}{3}+1\right] \quad \cdots\cdots$

④ Consider the adjoint Representation

its dimension $d[2, 1^{r-1}] = N^2 - 1$

According to $[T^a, T^b] = i f_{abc} T^c \Rightarrow (I^{ad}_{a})_{bc} = -i f_{a}$

Let's take $T^3$, and figure out its structure constant, and then write down $I^{ad}_{(T^3_{12})}$.

We have $[T^{(3)}_{12}, T^{(1)}_{12}] = i T^{(2)}_{12}$ and $[T^{(3)}_{12}, T^{(2)}_{12}] = -i T^{(1)}_{12}$, hence

$$\left(I^{ad}_{T^{(3)}_{12}}\right)_{T^{(1)}_{12} T^{(2)}_{12}} = i \quad \& \quad \left(I^{ad}_{T^{(3)}_{12}}\right)_{T^{(2)}_{12} T^{(1)}_{12}} = -i, \text{ i.e } \left(\begin{array}{ccc|c} 0 & -i & & \\ i & 0 & & \\ & & 0 & \\ \hline & & & \end{array}\right. \cdots$$

Then $[T^{(3)}_{12}, T^{(1)}_{1l}] = \frac{i}{2} T^{(2)}_{1l}$ $(l \geq 3)$ and $[T^{(3)}_{12}, T^{(2)}_{1l}] = -i/2\, T^{(1)}_{1l}$

$[T^{(3)}_{12}\ T^{(1)}_{2l}] = -i/2\, T^{(2)}_{2l}$ $(l \geq 3)$ $[T^{(3)}_{12}, T^{(2)}_{2l}] = \frac{i}{2} T^{(1)}_{2l}$

$$\Rightarrow \left(I^{ad}_{T^{(3)}_{12}}\right)_{T^{(1)}_{1l} T^{(2)}_{1l}} = i/2 \qquad \longrightarrow \qquad \left(\begin{array}{cc|cc} 0 & -i/2 & & \\ i/2 & 0 & & \\ \hline & & 0 & i/2 \\ & & -i/2 & 0 \end{array}\right)$$

$$\left(I^{ad}_{T^{(3)}_{12}}\right)_{T^{(1)}_{2l} T^{(2)}_{2l}} = -i/2$$

$$\text{Hence } \left(I^{ad}_{T^{(3)}_{12}}\right) = \left(\begin{array}{cc|c|ccc} 0 & -i & & & & \\ i & 0 & & & & \\ \hline & & 0 & & & \\ \hline & & & \begin{pmatrix} 0 & -i/2 \\ i/2 & 0 \end{pmatrix} & & \\ & & & & \begin{pmatrix} 0 & i/2 \\ -i/2 & \end{pmatrix} & \end{array}\right. \left.\begin{matrix} \\ \\ \\ \\ \end{matrix}\right\} \text{N-2 pairs}$$

Hence $$\mathrm{tr}\left[\ I^{ad}_{T^{(3)}_{12}}\ I^{ad}_{T^{(3)}_{12}}\ \right] = 1+1+\left(\tfrac{1}{4}+\tfrac{1}{4}+\tfrac{1}{4}+\tfrac{1}{4}\right)N-2$$

$$= N \qquad \text{or} \quad T_2\left[SU(N)_{ad}\right] = N$$

$$\Rightarrow \boxed{C_2\left[\,SU(N)\ \text{adjoint}\,\right] = \frac{g}{g}\,T_2 = N}$$

Consider an SU(2N) Hubbard model in the N→∞ limit, that each site has N-particles, then the Heisenberg model

$$H = J \sum_{\langle i,j\rangle} \sum_{a} I^a(i)\, I^a(j)$$ , where $I^a$ is defined for the

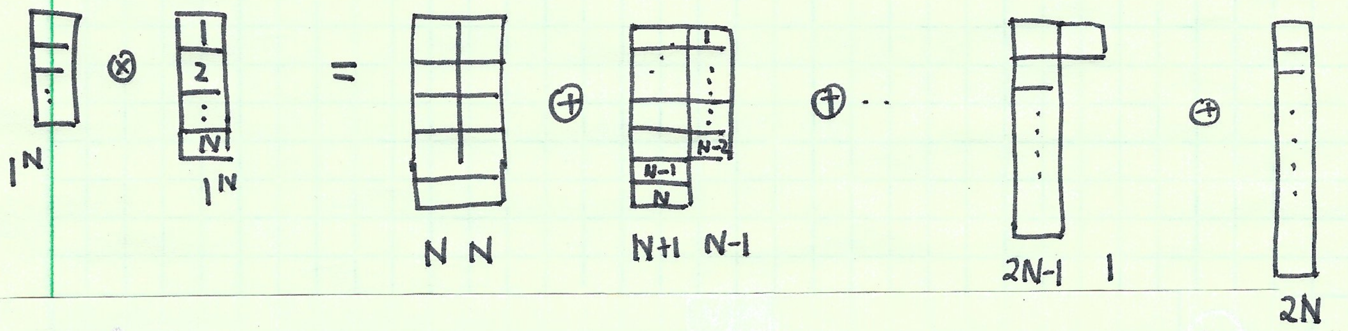
self-conjugate Reps  i.e $1^N$, and $a = 1, \cdots N^2-1$.

Just consider two-sites forming a bond, then

$$H_{ij} = \frac{J}{2}\left[\left[I^a(i) + I^a(j)\right]^2 - \sum_a I^{2,a}(i) - \sum_a I^{2,a}(j)\right]$$

we have $\sum_a I^{2a}(i) = \sum_a I^{2,a}(j) = \frac{N(2N-N)(2N+1)}{2\cdot(2N)} = \frac{N(2N+1)}{4}$

* Then we are facing a direct product decomposition problem

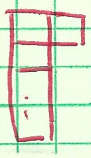


+1) . .

The ground state is a singlet, hence

$$\left(I^a(i) + I^a(j)\right)^2 = 0$$

$$E_G = -J\; N(2N+1)/4$$

For the first excited state , it's casimir = 2N

Hence $E - E_G = \frac{J}{2} \cdot 2N = NJ$