Lect 7 SU(3) group

§: Backgrounds:

① 3D harmonic oscillator $a_i = \frac{1}{2}\left(\frac{x}{l_0} + i P_0 \cdot l_0/\hbar\right)$ $a_i^\dagger \to a_j^\dagger u_{ji}$

$H = \sum_{i=1}^{3} \hbar\omega\, (a_i^\dagger a_i)$, which is invariant under $a_i \to (u^\dagger)_{ij} a_j$.

where $u^\dagger u = 1$, and u is 3×3 unitary matrix.

how many degrees of freedom: $9\times 2 - 3 - 3\times 2 = 9$

$$\left(\begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array}\right)\left(\begin{array}{ccc} | & | & | \end{array}\right) = \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{pmatrix}$$

9 complex elements; diagonal real; off-diagonal complex

If we further impose $\det u = 1$, then only 8 degrees of freedom.

Such a group $\{ u^\dagger u = 1, \text{ and } \det u = 1 \}$ is the SU(3) group.

EX: prove that there are N^2-1 degrees of freedom for SU(N) group.

② SU(3) group is widely used in high energy physics. — QCD.

quark has 3-colors R G B form a fundamental Rep of SU(3), i.e. $c_i^\dagger \to c_j^\dagger u_{ji}$ or $c_i \to u_{ij}^\dagger c_j$ $i = (R\ G\ B)$.

then $c_i \to c_j\, u_{ji}^{T,\dagger} = c_j (u^*)_{ji}$

hence $c^\dagger$ and c transform in fundamental and its complex conjugate — anti-fundamental Representations

They are represented as $\square$ and $\square^* = \begin{array}{|c|}\hline \\ \hline \\ \hline\end{array}$ (to be proved!)

Unlike the SU(2) case, there are non-equivalent to each other any more. $\square^*$ is the representation of anti-quarks.

- baryons are color-singlet — 3 quarks

consider a 3-quark state $|\psi\rangle = c_r^\dagger c_g^\dagger c_b^\dagger |\Omega\rangle$

under SU(3) transformation
$$|\psi\rangle \to c_i^\dagger c_j^\dagger c_k^\dagger\, u_{ri}\, u_{gj}\, u_{bk} |\Omega\rangle$$
$$= \left(\sum_P (-)^P u_{r\,P(r)}\, u_{g\,P(g)}\, u_{b\,P(b)}\right) c_r^\dagger c_g^\dagger c_b^\dagger |\Omega\rangle$$
$$= \det u \; c_r^\dagger c_g^\dagger c_b^\dagger |\Omega\rangle = c_r^\dagger c_g^\dagger c_b^\dagger |\Omega\rangle$$

Hence: n- particles form an SU(n) singlet

HW: prove that 2-quark states $|\psi_i\rangle = \epsilon_{ijk}\, c_j^\dagger c_k^\dagger |\Omega\rangle$ transform according to $\square^*$, i.e. $\square^* = \begin{array}{|c|}\hline \\ \hline \\ \hline\end{array}$.

§: Generators of SU(3) — Gell-man matrix

Consider infinitesimal transformation $u = 1 - i\epsilon T$, then

$u^\dagger u = 1 \Rightarrow \boxed{T^\dagger = T \text{ and } \operatorname{tr} T = 0.} \to$ Again this gives rise to $3^2 - 1 = 8$ generators.

Define operators $\hat{T} = c_i^\dagger T_{ij} c_j$, then

$$[H, T] = \hbar\omega [c_l^\dagger c_l, c_i^\dagger T_{ij} c_j] = \hbar\omega \{ c_l^\dagger T_{lj} c_j - c_i^\dagger T_{il} c_l \}$$

$$= \hbar\omega \, c_i^\dagger c_j [T_{ij} - T_{ij}] = 0 . \quad \text{— conserved quantities}$$

- We choose the bases matrices for T as (in the fundamental Rep.)

$$\boxed{\begin{gathered}
\lambda_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \lambda_2 = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \lambda_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
\lambda_4 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \lambda_5 = \begin{bmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{bmatrix} \quad \lambda_6 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \\
\lambda_7 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix} \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & -2 \end{bmatrix}
\end{gathered}}$$

They are normalized to $\mathrm{Tr}[\lambda_j \lambda_k] = 2\delta_{jk}$

we define the generators of the SU(3) group in the fundemental Rep as

$$\boxed{T^a = \frac{1}{2} c_i^\dagger \lambda^a_{ij} c_j \qquad a = 1 \sim 8}$$

$$\boxed{\text{HW: prove the casimir } \sum_{a=1}^{8} (T^a)^2 = \frac{4}{3}}$$ ← for the fundementa[l] Rep.

⊛ The adjoint Representation

$$[T^a, T^b] = \sum_c T^c \left(I^{ad}_a\right)_{cb} \leftarrow u\, T^b u^{-1} = \Sigma\, T^c D^{ad}_{cb}$$

Hence I^{ad}_a for $a = 1 \sim 8$ is just the structure constant of $SU(3)$

Compare $[T^a, T^b] = i f_{abc} T^c$, where have

$$\left(I^{ad}_a\right)_{cb} = i f_{abc} \quad \text{or} \quad \boxed{\left(I^{ad}_a\right)_{bc} = -i f_{abc}}$$

HW: test the structure constants are anti-symmetric

ABC	123	147	156	246	257	345	367	458	678
f_{ABC}	1	½	-½	½	½	½	-½	√3/2	√3/2

write down the 8x8 matrix for $I^{ad}_1 \sim I^{ad}_8$.

For example:

$$I^{ad}_1 = \begin{array}{cc} \begin{bmatrix} 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 0 & -i & \cdot & \cdot & \cdot & \cdot & \cdot \\ & i & 0 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 0 & \cdot & \cdot & -\tfrac{1}{2}i & \cdot \\ & & & & 0 & \tfrac{1}{2}i & & \\ & & & & -\tfrac{1}{2} & 0 & & \\ & & & \tfrac{1}{2}i & \cdot & \cdot & 0 & \\ & & & & & & & 0 \end{bmatrix} & \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{matrix} \end{array}$$

What's the Casimir of the adjoint Rep ?

Since $T_{jk} = \mathrm{Tr}\,[\,I_j^{ad}\, I_k^{ad}\,] = \delta_{jk}\, T_2(ad)$

set $j=k$, and sum over j $\Rightarrow$ $\mathrm{Tr}\,[(I_j^{ad})^2\,] = T_2(ad)\cdot g$

$$= C_2(ad)\cdot M_{ad}$$

Since $M_{ad} = g$, we have $\boxed{C_2(ad) = T_2(ad)}$

Take $j=k=1$ $\Rightarrow$ $\mathrm{Tr}[\,I_1^2\,] =$ $(1+1+1/4+1/4+1/4+1/4) = 3$

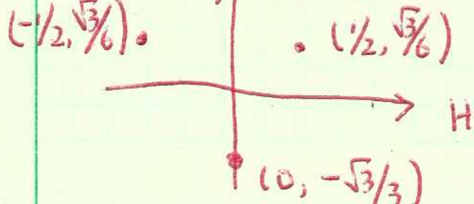
$\Rightarrow$ $\boxed{C_2(ad) = 3 \text{ for } SU(3) \text{ adjoint}}$

(*) Weights and roots

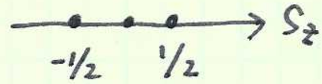
Among the 8-generators of the SU(3) group, T_3 and T_8 commut with each other, They play the same role of S_z for the SU(2) group. Th the fundment representation can be expressed as a lattice in th 2D plane according to their eigenvalues of (T_3, T_8).

$$\begin{pmatrix}1\\0\\0\end{pmatrix} \rightarrow (1/2, \sqrt{3}/6) \qquad \begin{pmatrix}0\\1\\0\end{pmatrix} \rightarrow (-1/2, \sqrt{3}/6) \qquad \begin{pmatrix}0\\0\\1\end{pmatrix} \rightarrow (0, -\sqrt{3}/3)$$

with $H_1 = T_3 = \frac{1}{2}\begin{pmatrix}1 & & \\ & -1 & \\ & & 0\end{pmatrix}$, $H_2 = T_8 = \frac{\sqrt{3}}{6}\begin{pmatrix}1 & & \\ & 1 & \\ & & -2\end{pmatrix}$, i.e.

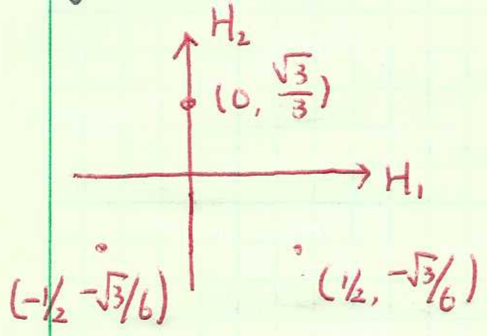


c.f.

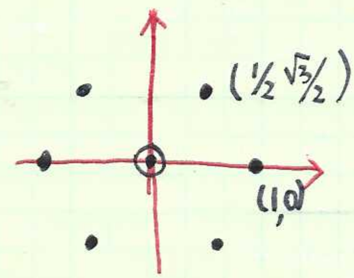


As fir the anti-fundement Rep, since $\mathcal{U} = e^{iM} \longrightarrow \mathcal{U}^* = e^{i(-M^*)}$

hence, its generaters are $-T_i^*$ ($i = 1, \cdots 8$). Sine T_3 and T_8 are real, $H_1 = -T_1$ and $H_2 = -T_3$. Hence the weight diagram of the anti-fundamental Rep is



HW: Basing on the matrices in the adjoint Rep, diagonize it, and plot the weight diagram for the adjoint representation.



~~⊗ roots (weight of the adjoint Rep)~~

⊗ The canonical firm of simple Lie algebra, — Cartan subalgebra and roots.

Consider the adjoint Rep. Its generators satisfy $\boxed{tr[I_i^{(ad)} I_j^{(ad)}] = \lambda\delta}$

For each vecter of the Lie algebra, we define $|A\rangle \leftrightarrow I_A$, and (7)

the inner product is defined as

$$\langle A|B\rangle \equiv \lambda^{-1}\, tr[A^{\dagger(ad)} B^{(ad)}]$$

- The above inner product satisfies

$$\langle A|(C_1 B + C_2 D)\rangle = C_1\langle A|B\rangle + C_2\langle A|D\rangle$$

$$\langle C_1 A + C_2 D|B\rangle = C_1^*\langle A|B\rangle + C_2^*\langle D|B\rangle$$

$$\langle A|B\rangle = \langle B|A\rangle^*,$$

we also define $D|B\rangle = |[D, B]\rangle$, then

$$\langle A|D|B\rangle = \lambda^{-1}\, tr[A^{\dagger(ad)} D^{(ad)} B^{(ad)} - A^{\dagger(ad)} B^{(ad)} D^{(ad)}]$$

$$= \lambda^{-1}\, tr[[A^{\dagger(ad)}, D^{(ad)}], B^{ad}] = \lambda^{-1}\, tr[[D^{\dagger(ad)}, A^{(ad)}]^{\dagger}, B^{ad}]$$

hence $\boxed{\langle A|D|B\rangle = \langle [D^\dagger, A]|B\rangle}$

Any vecter is also a liner operater : $X|Y\rangle = |[X, Y]\rangle$. Hence we can calculate the eigenvalue and eigenvecters of X. For a Lie algebra, there exist l commutable vecters

$$\boxed{[H_i, H_j] = 0, \quad 1 \le i, j \le l.}$$

They span a linear sub-algebra, which is called Cartan subalgebra.

For $SU(2)$, rank-1, $H = S_z$

$SU(3)$ rank-2 $H = \{T_3, T_8\}$

In the remaining of the algebra, there exist $g-l$ vectors.

They can be organized as common eigenvectors of $\{H_i\}$.

$$\boxed{H_j\,|E_\alpha\rangle = \alpha_j\,|E_\alpha\rangle, \text{ or } [H_j, E_\alpha] = \alpha_j E_\alpha .}$$

where α_j is not a zero vector, and $\vec{\alpha}$ does not repeat.

These $|E_\alpha\rangle$ are called roots, and $(\alpha_1 \cdots \alpha_l) = \alpha_j$ is root vector.

we normalize
$$\boxed{\begin{aligned} \langle H_i | H_j \rangle &= \lambda^{-1}\, tr[\, I_i^{(ad)} I_j^{(ad)}] = \delta_{ij} \\ \langle E_\alpha | E_\beta \rangle &= \lambda^{-1}\, tr[\, E_\alpha^{\dagger(ad)} E_\beta^{(ad)}] = \delta_{\alpha\beta} \end{aligned}}$$

Example: $su(2)$ $\quad H = S_z$, $\quad E_\pm = \frac{1}{\sqrt{2}}(S_x \pm i S_y)$

$$[H, E_\pm] = \pm E_\pm$$

From Jocabi identity,
$$\begin{aligned} 0 &= [H_j, [E_\alpha, E_\beta]] + [E_\alpha, [E_\beta, H_j]] + [E_\beta, [H_j, E_\alpha]] \\ &= [H_j, [E_\alpha, E_\beta]] + (\alpha_j + \beta_j)[E_\beta, E_\alpha] \end{aligned}$$

$\Rightarrow [H_j, [E_\alpha, E_\beta]] = (\alpha_j + \beta_j)[E_\alpha, E_\beta]$, hence

$$[E_\alpha, E_\beta] = \begin{cases} N_{\alpha,\beta}\, E_{\alpha+\beta} & \text{if } \alpha+\beta \text{ is a root} \\ \sum \lambda_j H_j & \text{if } \alpha+\beta = 0 \\ 0 & \text{if } \alpha+\beta \text{ is not a root} \end{cases}$$

HW: Prove that $\lambda_j = \alpha_j$. i.e $[E_\alpha, E_{-\alpha}] = \alpha_j H_j$

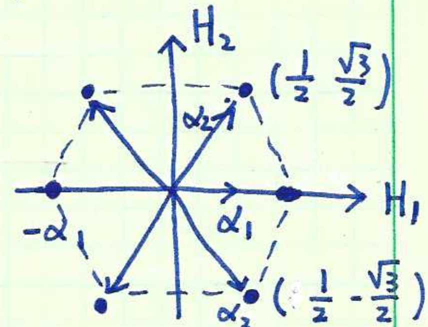
Hint: check $\langle H_j | E_\alpha | E_{-\alpha} \rangle$

HW: check for SU(3), the roots can be organized as

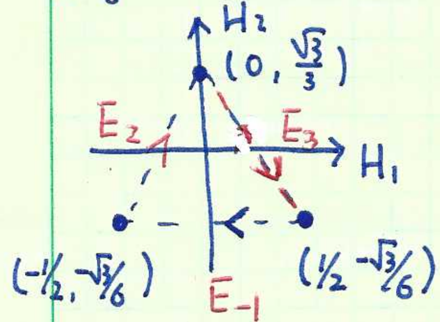
$$E_{\pm 1} = \frac{1}{\sqrt{2}}(T_1 \pm i T_2) \qquad \pm\alpha_1 = (\pm 1, 0)$$

$$E_{\pm 2} = \frac{1}{\sqrt{2}}(T_4 \pm i T_5) \qquad \pm\alpha_2 = \pm(\tfrac{1}{2}, \tfrac{\sqrt{3}}{2})$$

$$E_{\pm 3} = \frac{1}{\sqrt{2}}(T_6 \mp i T_7) \qquad \pm\alpha_3 = (\pm\tfrac{1}{2}, \mp\tfrac{\sqrt{3}}{2})$$

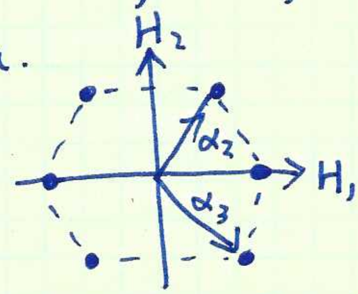


- Apply roots to the weight diagram



$E_{\pm 1}$, $E_{\pm 2}$, $E_{\pm 3}$

are generalizations of $S_\pm$ for SU(2) algerbra.



Ⓐ positive weight / root

For a weight / root diagram, if a eigenvalue vecter $\{\alpha_i\}$, it's first non-zero value > 0, then it's called positive weight / root.

Simple roots are positive roots that cannot be describeed by a sum of other positive roots. Any positive root can be represent as a sum of simple roots with non-negative coefficents.

For example, α_2, α_3 are simple roots of SU(3) algebra.