Phy 220 Final exam Jun 2020

We study the SU(4) group and its applications. Its Cartan subalgebra

$$T_1^{(3)} = \frac{1}{2}\begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}, \quad T_2^{(3)} = \frac{1}{\sqrt{12}}\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -2 & \\ & & & 0 \end{pmatrix}, \quad T_3^{(3)} = \frac{1}{\sqrt{24}}\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -3 \end{pmatrix}.$$

① Draw the Dykin digram of SU(4) and explain the meanings of the symbols,

② Denote the simple roots as

$$\alpha^1 = (1, 0, 0), \quad \alpha^2 = (-\tfrac{1}{2}, \tfrac{\sqrt{3}}{2}, 0), \quad \alpha^3 = (0, -\sqrt{\tfrac{1}{3}}, \sqrt{\tfrac{2}{3}})$$

(Here, we use the convention of positive roots by counting non-negative components from right to left).

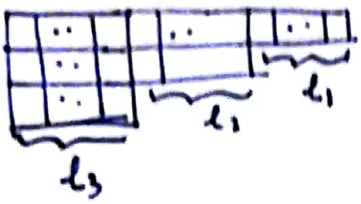
Construct the Cartan Matrix for the SU(4). And Based on the Cartan matrix, construct all other root vectors.

(You donot need to work out their commutation rules).

③ Work out the fundement weights $\vec{\mu}_1, \vec{\mu}_2, \vec{\mu}_3$. Calculate the dimensions and the Casimirs of the representations $\vec{\mu}_1, \vec{\mu}_2, \vec{\mu}_3$.

(Acctually, they correspond to the Young patterns $\square$, $\begin{array}{|c|}\hline \; \\ \hline \; \\ \hline \end{array}$, $\begin{array}{|c|}\hline \; \\ \hline \; \\ \hline \; \\ \hline \end{array}$, respectively).

4 You don't need to prove: The representation of the weight $\vec{\mu} = l_1\vec{\mu}_1 + l_2\vec{\mu}_2 + l_3\vec{\mu}_3$ denoted as (l_1, l_2, l_3), corresponds to



i What are the dimensions and the value of Casimir of the representation (l_1, l_2, l_3)? (You can use the knowledge learned in Lect 15).

Consider the fundamental spinor Rep of the SU(4). Its σ_1-type generators $(T^{(1)}_{ab})_{cd} = \frac{1}{2}(\delta_{ac}\delta_{bd} + \delta_{bc}\delta_{ad})$, and its σ_2-type generators $(T^{(2)}_{ab})_{cd} = \frac{-i}{2}(\delta_{ac}\delta_{bd} - \delta_{bc}\delta_{ad})$ where $1 \le a < b \le 4$.

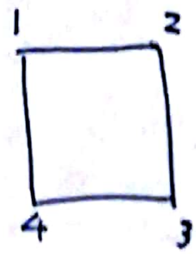
The σ_3-type matrices $(T^{(3)}_{a})_{cd}$ are defined in page 1, $1 \le a \le 3$.

5. As a warming up, please

Conside a 4-site plaquette on each site. We define an SU(2) spin-1/2 Heisenberg model as

$$H = J \sum_{\langle ij \rangle} \vec{S}_i \cdot \vec{S}_j,$$

where $\langle ij \rangle$ represents neighbouring sites. This Hamiltonican can be solved by the following trick:



$$H = J/2\left[C_2(1234) - C_2(13) - C_2(24)\right]$$

where $C_2(13) = \vec{S}_{13}\cdot\vec{S}_{13}$ with $\vec{S}_{13} = \vec{S}_1 + \vec{S}_3$ – the total spin of $\vec{S}_1$ and $\vec{S}_3$; $C_2(24) = \vec{S}_{24}\cdot\vec{S}_{24}$ with $\vec{S}_{24} = \vec{S}_2 + \vec{S}_4$.

$C_2(1234) = \vec{S}_{tot}\cdot\vec{S}_{tot}$ with $\vec{S}_{tot} = \vec{S}_1 + \vec{S}_2 + \vec{S}_3 + \vec{S}_4$.

There are 2^4=16 states in the Hilbert space, please classify all the eigenstates of H in terms of SU(2) representations, and calculate their eigen-energies.

6. We consider its SU(4) version, and on each site, there is a SU(4) fundamental spinor quartet. We define the SU(4) Heisenberg model as

$$H = J\sum_{\langle ij\rangle}\left[\sum_{1\le a<b\le 4} T^{(1)}_{ab}(i)\,T^{(1)}_{ab}(j) + T^{(2)}_{ab}(i)\,T^{(2)}_{ab}(j) + \sum_{1\le a\le 3} T^{(3)}_{a}(i)\,T^{(3)}_{b}(j)\right].$$

You can use a similar trick to the SU(2) case to solve this problem. — You can contruct the SU(4) version of $C_2(1,3)$, $C_2(2,4)$, and $C_2(1234)$. You can diagonalize $C_2(13)$ and $C_2(2,4)$ by decomposing the direct product $\square \times \square$.

Then you can diagonalize $C_2(1234)$ by decomposing

$$(\square \times \square) \times (\square \times \square).$$

There are $4^4 = 256$ states in the Hilbert space. Solve each eigen-energy including the ground state and all excited states. Mark the Young pattern of each set of eigenstates. Figure out the dimensions, i.e. the degenercies, of each set of eigenstates.