Lect 2: Basic ideas of group

§ Definition:

A group G is a set of elements $\{g_i\}$. on which the multiplication of two elements is defined. The product $g_3 = g_1 g_2$ also belongs to G. The product may not be exchangable. The product obeys the following rules

① Associativity $g_1(g_2 g_3) = (g_1 g_2) g_3$

② Existence of an identity $e \in G$, such that for all $g \in G$,

$$eg = g.$$

③ Existence of inverse: for each $g \in G$, there exists an elements of g^{-1}, such that $g^{-1} g = e$

Comment ① typically, g_i can be represented as symmetry operations

② G can be finite group, or infinite group. For infinit groups, they can be discrete/countable (e.g. braid group), or continuous group (Lie group)

③ Excerise: prove ① $ge = g$ ② $gg^{-1} = e$

i.e. e and g, g and g^{-1} are always exchangable.

④ Excercise: prove that the identity is unique, the inverse of g is also unique.

§ Examples of groups

- Real numbers form a group. The product of group elements is just the ordinary addition.

$$x, y \in R \rightarrow x+y \in R, \quad x+0=x, \quad x+(-x)=0.$$

- Real numbers excluding zero form a group. The product is the ordinary product $\quad x \cdot 1 = x, \quad x \cdot \frac{1}{x} = 1.$

- Finite groups

① order 2 : Z_2 mutiplication table

	e	σ
e	e	σ
σ	σ	e

σ: parity operation, or reflection, or rotation 180°, etc

② order 3 : Z_3

	e	ω	ω'
e	e	ω	ω'
ω	ω	ω'	e
ω'	ω'	e	ω

· cyclic group of order 3 $\rightarrow \{e, \omega, \omega^2\}$

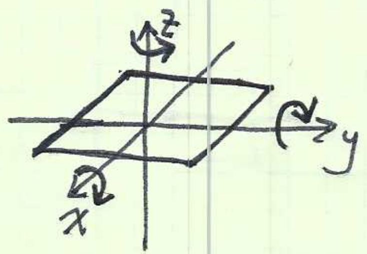
Abeliean group, $g_i g_j = g_j g_i$

Rotation around z-axis : $0°, \pm 120°$.

Z_n: $\{e, \omega^1, \omega^2, \cdots \omega^{n-1}\}$, where ω represents rotation at $2\pi/n$.
or C_n

③ order of 4 : Z_4, or $D_2 = Z_2 \otimes Z_2$

$R_{x,y,z}$: rotations around x, y, z axes at π



	e	R_x	R_y	R_z
e	e	R_x	R_y	R_z
R_x	R_x	e	R_z	R_y
R_y	R_y	R_z	e	R_x
R_z	R_z	R_y	R_x	e

☆ its still an abeliean group

☆ it can be decomposed to direct product of two Z_2's.

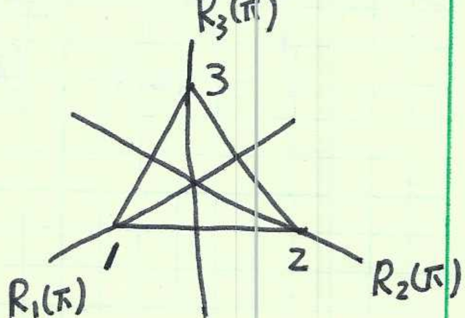
$\{e, R_x\} \otimes \{e, R_y\} = \{e, R_x, R_y, R_z\}$

D_2 can be generalized to D_n — dihedral group

④ order of 5 : Z_5

⑤ order of 6 : $Z_2 \otimes Z_3 = Z_6$, D_3 — the smallest non-abelian group

ω : $R_z(\frac{2}{3}\pi)$, ω^2 : $R_z(\frac{4}{3}\pi)$



	e	ω	ω^2	R_1	R_2	R_3
e	e	ω	ω^2	R_1	R_2	R_3
ω	ω	ω^2	e	R_3	R_1	R_2
ω^2	ω^2	e	ω	R_2	R_3	R_1
R_1	R_1	R_2	R_3	e	ω	ω^2
R_2	R_2	R_3	R_1	ω^2	e	ω
R_3	R_3	R_1	R_2	ω	ω^2	e

D_3 is isomorphic to permutation group S_3

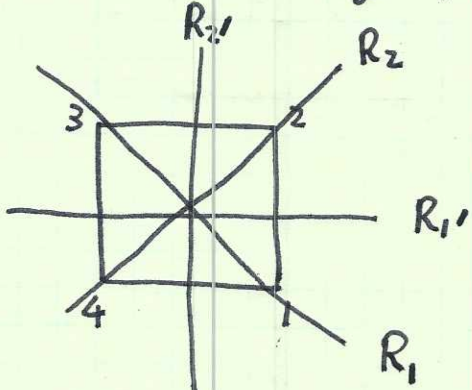
$e: \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$, $\omega \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$ $\omega^2 \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$

$R_1: \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$ $R_2 \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$ $R_3 \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$

6 order of 7 $\rightarrow Z_7$

7 order of 8 : Z_8, $Z_2 \otimes Z_4$, $Z_2 \otimes Z_2 \otimes Z_2$ (they are not equal)

- non-abelieean D_4 (symmetry of square)



$R_{1,2}(\pi)$ pass the vertices, i.e. diagonal

$R_{1',2'}(\pi)$: bisectors

$\omega: R_z(\frac{\pi}{2})$

	e	ω	ω^2	ω^3	R_1	R_2	R_1'	R_2'
e	e	ω	ω^2	ω^3	R_1	R_2	R_1'	R_2'
ω	ω	ω^2	ω^3	e	R_1'	R_2'	R_2	R_1
ω^2	ω^2	ω^3	e	ω	R_2	R_1	R_2'	R_1'
ω^3	ω^3	e	ω	ω^2	R_2'	R_1'	R_1	R_2
R_1	R_1	R_2'	R_2	R_1'	e	ω^2	ω^3	ω'
R_2	R_2	R_1'	R_1	R_2'	ω^2	e	ω'	ω^3
R_1'	R_1'	R_1	R_2'	R_2	ω'	ω^3	e	ω^2
R_2'	R_2'	R_2	R_1'	R_1	ω^3	ω'	ω^2	e

Q : quarterniun group $\qquad g = a + ib + jc + kd$

$i^2 = j^2 = k^2 = -1, \quad ij = -ji = k,$

set $-1 = e'$
$-i = i'$
$-j = j'$
$-k = k'$

	e	i	j	k	e'	i'	j'	k'
e	e	i	j	k	e'	i'	j'	k'
i	i	e'	k	j'	i'	e	k'	j
j	j	k'	e'	i	j'	k	e	i'
k	k	j	i'	e'	k'	j'	i	e
e'	e'	i'	j'	k'	e	i	j	k
i'	i'	e	k'	j	i	e'	k	j'
j'	j'	k	e	i'	j	k'	e'	i
k'	k'	j'	i	e	k	j	i'	e'

They are the double group of D_2. We can represent them as rotatiun of spin-1/2 particle.

e – rotatiun angle 0

e' – rotatiun angle 2π

i – rotatiun around x-axis at angle π

$$e^{-i\sigma_x \frac{\pi}{2}} = -i\sigma_x$$

i' – rotatiun around x-axis at angle $-\pi$

$$e^{-i\sigma_x (-\frac{\pi}{2})} = i\sigma_x$$

$j \to -i\sigma_y \quad , \quad j' \to i\sigma_y$

$k \to -i\sigma_z \qquad k' \to i\sigma_z$

§ Basic concepts

- Subgroup: if a subset of a group G also forms a group, it is called a subgroup.

Example: Z_2 is a subgroup of D_2, and D_2 is a subgroup of D_4.

- Cosets: Consider a subgroup $H \subseteq G$ with elements $\{h_1, h_2 \cdots\}$. For an element $g \in G$, we form the (left) coset $gH = \{gh_1, gh_2, \cdots\}$.

★ for two cosets g_1H and g_2H, they either have $g_1H = g_2H$, or they do not share any common elements.

Proof: if $g_1h_1 = g_2h_2 \Rightarrow g_2 = g_1(h_1h_2^{-1})$, hence for any h_i, we have $g_2h_i = g_1(h_1h_2^{-1}h_i) \Rightarrow g_1H = g_2H$.

★ If H is a finite group, each coset has the same number of distinct elements as H.

★ If G is also a finite group, then G can be decomposed into the sum of cosets

$$G = g_1H + g_2H + \cdots g_nH_n$$

⇒ (Lagrange's theorem) The order of H must divide the order of G.

⊛ Congugation and congugacy class

G is a group with elements $\{g_a\}$. The congugate of any element g_a with respect to any other group element g as

$$\tilde{g}_a = g\, g_a\, g^{-1}.$$

- Such a transformation leaves the multiplication table into itself.

$$g_a g_b = g_c \longrightarrow \tilde{g}_a \tilde{g}_b = \tilde{g}_c$$

Class: We organize a group G into non-overlapping sets by using conjugacy class.

Pick up an element g_a, conjugate it with respect to all elements,

→ generate a set of conjugacy class of g_a

$$\boxed{C_a : \quad \tilde{g}_a = g\, g_a\, g^{-1}, \quad \forall\, g \in G.}$$

For C_a and C_b generated by two different elements g_a and g_b, they donot have common elements, otherwise they are identical.

⊛ Each element in an Abelian group is a class.

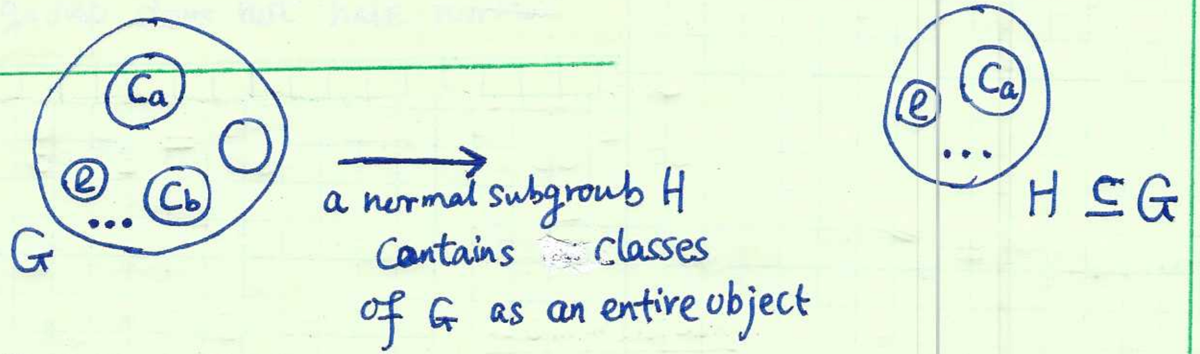
⊛ The identity e is a class by itself.

⊛ Normal subgroup ~~and~~ / simple group / quotient group

For a subgroup $H \subseteq G$, if we conjugate h_i with respect to any g in G, and

$$g h_i g^{-1} \equiv h_j \in H, \quad \forall g \in G,$$

the H is called the normal subgroup.



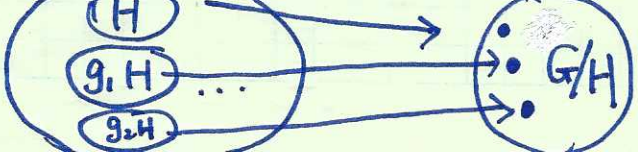
For a normal subgroup H, we define the $\overbrace{\text{cosets}}^{}$ of G (set of)

$G/H \equiv \{g_1 H, g_2 H, \cdots\}$. We define a multiplication rule on the set of cosets: by taking a representive element from each of $g_i H$ and $g_j H$, (say, $g_i h_a$ and $g_j h_b$) and define $(g_i H)(g_j H)$ to be the coset that $g_i h_a g_j h_b$ lies.

More precisely, $g_i h_a g_j h_b = g_i g_j \underbrace{(g_j^{-1} h_a g_j)} h_b = g_i g_j h_c h_b$

$$= g_i g_j h_e \in g_i g_j H.$$

Then G/H is defined as the quotient group.



factorization of G.

If a group G does not have a normal subgroup H, expcet the trivil case that H = {e}, G, then G is called the simple group.

Remark: a normal group is like a factor of G, and a simple group is like a prime number.

Example ★ in 3D rotation group SO(3), the conjugacy classes are the sets of rotations at the same angle, but about different axes.

★ For the group $\mathcal{U}(N)$, $N \times N$ unitary matrices, the conjugacy classes at the sets of matrices possessing the same eigenvalues

★ permutation group: Each permuation can be represented as a series of cycles without common numbers. For example

$$\pi_1 : \begin{pmatrix} 1&2&3&4&5&6&7&8 \\ 2&3&1&5&4&7&6&8 \end{pmatrix} = (123)(45)(67)(8)$$

$$\pi_2 : \begin{pmatrix} 1&2&3&4&5&6&7&8 \\ 2&3&4&5&6&7&8&1 \end{pmatrix} = (12345678)$$

$$\pi_1 \cdot \pi_2 = \begin{pmatrix} 2&3&4&5&6&7&8&1 \\ 3&1&5&4&7&6&8&2 \end{pmatrix} \begin{pmatrix} 1&2&3&4&5&6&7&8 \\ 2&3&4&5&6&7&8&1 \end{pmatrix} = \begin{pmatrix} 1&2&3&4&5&6&7&8 \\ 3&1&5&4&7&6&8&2 \end{pmatrix}$$

$$= (135782)(4)(6)$$

★ Each class consists all permutations with the same cycle pattern

$$\underbrace{(*)(*)\cdots(*)}_{r_1}\quad\underbrace{(**)\cdots(**)}_{r_2}\quad\cdots\quad\underbrace{(\underbrace{**\cdots}_{m})\cdots(\underbrace{***\cdot}_{m})}_{r_m}\quad\cdots$$

This conjugacy class is denoted by $(r_1, r_2, \cdots r_n)$, with $r_j \geq 0$.

$$\boxed{r_1 + 2r_2 + \cdots nr_n = n} \quad \text{for } S_n$$

Each class has # of elements $\dfrac{n!}{1^{r_1}(r_1!)\,2^{r_2}(r_2!)\cdots n^{r_n}(r_n!)}$.

★ For S_n, how many different classes?

define $\lambda_1 = r_1 + \cdots r_n$

$\lambda_2 = r_2 + \cdots r_n$

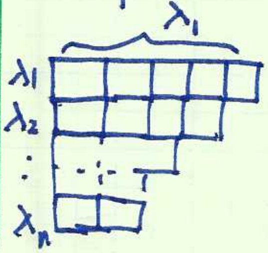
$\vdots$

$\lambda_n = r_n$

$\Rightarrow \lambda_1 + \cdots \lambda_n = n$

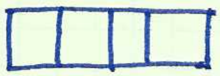
and $\lambda_1 \geq \lambda_2 \geq \cdots \lambda_n$

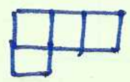
represent the partition $\{\lambda_1, \cdots \lambda_n\}$ as

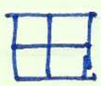


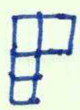
Example S_4: $(X)(X)(X)(X) \to r_1 = 4,\ r_2 = r_3 = r_4 = 0$

$\Rightarrow \{\lambda_1 \cdots \lambda_4\} = \{4, 0, 0, 0\}$



$(XX)(X)(X)$ $\quad r_1 = 2,\ r_2 = 1,\ r_3 = r_4 = 0 \Rightarrow \{3, 1, 0\,0\} \to$ 

$(XX)(XX)$ $\quad r_1 = 0\ \ r_2 = 2,\ r_3 = r_4 = 0 \Rightarrow \{2, 2, 0\,0\} \to$ 

$(XXX)(X)$ $\quad r_1 = 1\ \ r_2 = 0\ \ r_3 = 1,\ r_4 = 0 \Rightarrow \{2, 1, 1, 0\} \to$ 

$(XXXX)$ $\quad r_1 = r_2 = r_3 = 0,\ r_4 = 1 \Rightarrow \{1\,1\,1\,1\} \to$