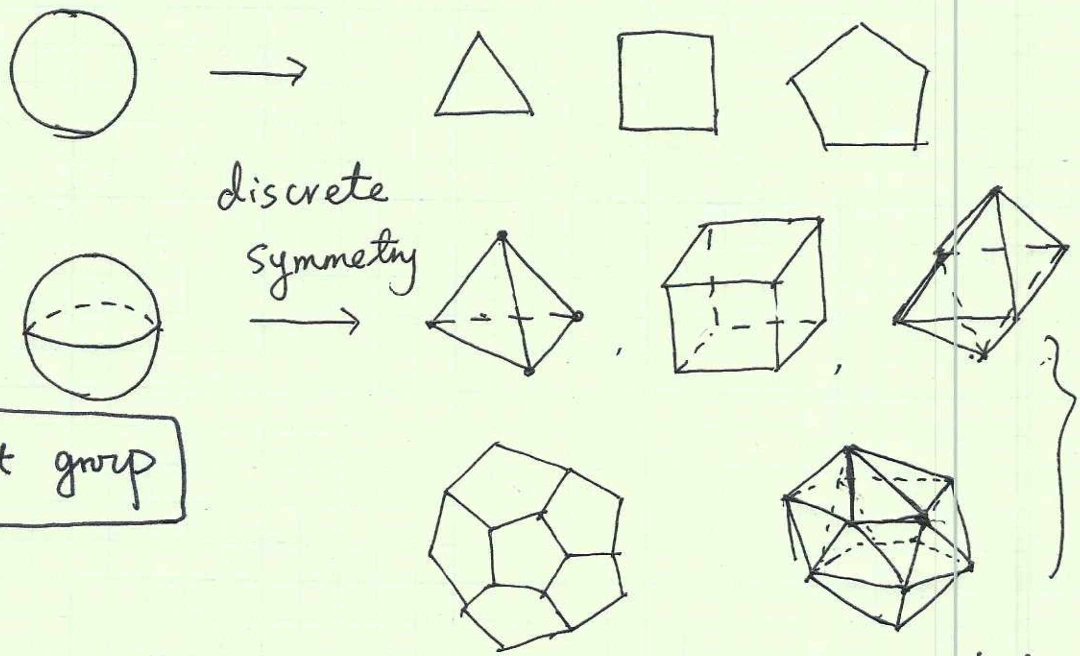
Lect 1: The symmetry principle, group fundamentals

§: Examples in daily life — Weyl: "Symmetry"

- 

discrete symmetry

$\boxed{\text{point group}}$

Platonic solids

dodecahedron   icosahedron ↳ C_{60}

- crystal: 7 - crystalline systems
  14 - Bravis lattice
  32 - crystalline point group
  230 3 dimensional crystals — $\boxed{\text{space group}}$

discrete translational symmetry

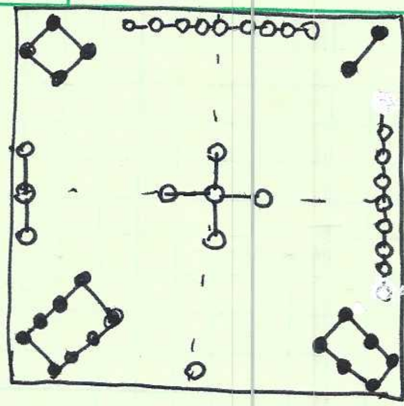
- log - spiral   fixed angle

$$\rho = c\, e^{b\theta}$$

"Eadem mutata resurgo" – J. Bernou li

$\boxed{\text{rotation + scaling} \longrightarrow \text{invariance}}$

" Why does a moth fly into a flame?"



⊛ Magic square (Luo-shu)

⊛ Viéte theorem – $\boxed{\text{permutation group}}$

symmetric polynomials

$$x^2 + a_1 x + a_2 = 0 \Rightarrow \begin{cases} x_1 + x_2 = -a_1 \\ x_1 x_2 = a_2 \end{cases}$$ invariant under permutation (12)

how about $x_1 - x_2$? → odd under (1 2)

hence $(x_1 - x_2)^2$ is invaiant $\Rightarrow (x_1 - x_2)^2 = (x_1 + x_2)^2 - 4x_1x_2 = a_1^2 - 4a_2$

$$\Rightarrow \begin{cases} x_1 + x_2 = -a_1 \\ x_1 - x_2 = \sqrt{\Delta}, \text{ with } \Delta = a_1^2 - 4a_2 \end{cases} \Rightarrow x_{1,2} = \frac{-a_1 \pm \sqrt{\Delta}}{2}$$

$x_1 \pm x_2$: reprensentations of the permutation group S_2, or Z_2

even: A_1

odd: A_2

→ Lagrange : symmetry and radical formulae

Galois theory : under what conditions, an equation $x^n + a_1 x^{n-1} + \cdots a_n = 0$ can be solvable by radicals.

In HW, you will use the property of S_3 to find the radical formula for a cubic equation.

n- permutation group S_n : $\begin{pmatrix} 1 & 2, & \cdots & n \\ a_1 & a_2 & \cdots & a_n \end{pmatrix}$, $n!$

$x^3 + px + q = 0$: define $\Delta = \left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3$, $\omega = e^{i\frac{2}{3}\pi}$

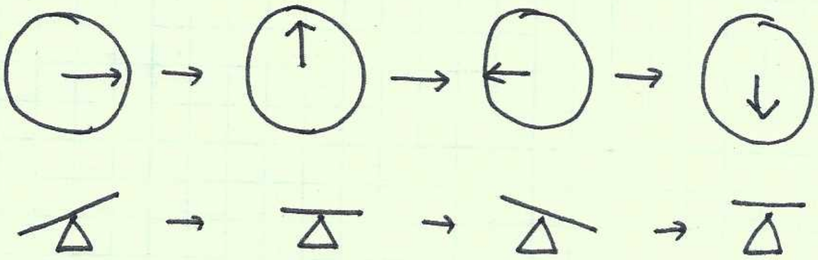
$$u = \left(-\frac{q}{2} + \sqrt{\Delta}\right)^{1/3}, \quad v = \left(-\frac{q}{2} - \sqrt{\Delta}\right)^{1/3}$$

$$\Rightarrow \begin{cases} x_1 = u + v \\ x_2 = u\omega + v\omega^2 \\ x_3 = u\omega^2 + v\omega \end{cases} \leftarrow \text{Cardano formula.}$$

(*) time-involved symmetries (S. Xu, C. Wu, PRL 120, 096401 (2018))

Clock

see-saw



$\boxed{\text{space-time group}}$

§ Examples : space-time symmetry

- time translation symmetry : $\frac{dH}{dt} = -\frac{\partial L}{\partial t} = 0 \rightarrow$ energy conservation
- space translation symmetry
  → momentum conservation $\quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = \frac{\partial L}{\partial q} = 0$
  if L doesn't depend on q
- space isotropy ⟶ angular momentum conservation

$$SO(3) \xrightarrow{c=1} \text{Lorentz group } SO(3,1) \longrightarrow \text{Poincáre group}$$

$$SO(3) \xrightarrow{c=\infty} \text{Galilean group}$$

☆ SO(n), SU(n), Sp(2n) ④

- Harmonic oscillators $H = \left[\left(\frac{x}{l_0}\right)^2 + \left(\frac{P \cdot l_0}{\hbar}\right)^2\right] \frac{\hbar\omega}{2}$

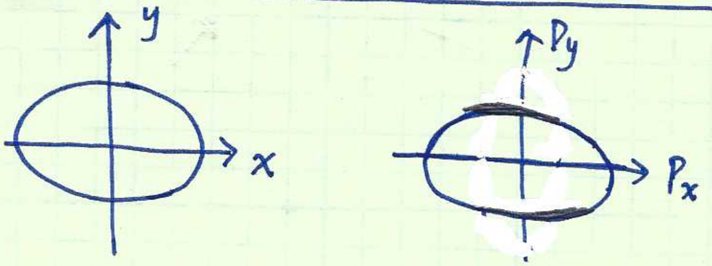
$$a_x = \frac{1}{\sqrt{2}}\left[x/l_0 + i P \cdot l_0/\hbar \right] \qquad a_x^\dagger = \frac{1}{\sqrt{2}}\left[x/l_0 - i P l_0/\hbar \right]$$

2D harmonic oscillator $H = \frac{\hbar\omega}{2} (a_x^\dagger \; a_y^\dagger) \begin{pmatrix} a_x \\ a_y \end{pmatrix}$

H is invariant under SU(2) transformation $\begin{pmatrix} a_x' \\ a_y' \end{pmatrix} \to u \begin{pmatrix} a_x \\ a_y \end{pmatrix}$.

u's generator $\tau_x = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$, $\tau_y = \begin{pmatrix} & -i \\ i & \end{pmatrix}$, $\tau_z = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$

$$\Rightarrow \boxed{\begin{aligned} \hbar\tau_y &= \frac{-i}{\hbar}(a_x^\dagger a_y - a_y^\dagger a_x) = x P_y - y P_x \leftarrow \text{2D angular momentum} \\ \hbar\tau_x &= \hbar[a_x^\dagger a_y + a_y^\dagger a_x] = \hbar\left[\frac{xy}{l_0^2} + \frac{P_x P_y l_0^2}{\hbar^2}\right] \\ \hbar\tau_z &= \qquad = \frac{\hbar}{2}\left[\frac{x^2 - y^2}{l_0^2} + \frac{(P_x^2 - P_y^2) l_0^2}{\hbar}\right] \end{aligned}}$$



Quadrapole

3D → SU(3) symmetry

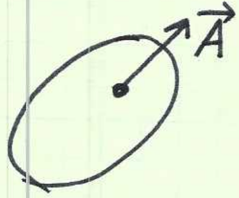
nD → SU(N) symmetry

} symmetry group widely used in high energy physics.

- Kepler problem — hydrogen problem

angular momentum conservation $\longrightarrow$ planar orbit / $2l+1$ fold degeneracy

Runge – Lentz vector $\longrightarrow$ eliptical orbit



$$\vec{L}\,, \quad \vec{A} = \frac{1}{m\gamma}\,\vec{P}\times\vec{L} - \hat{r}$$

$SO(4)$ symmetry $\rightarrow$ n^2 degeneracy

- Symplectic transformation (canonical transformation)

$$q = q(Q, \mathbb{P}) \qquad \text{define } M = \begin{pmatrix} \frac{\partial Q}{\partial q} & \frac{\partial Q}{\partial p} \\ \frac{\partial \mathbb{P}}{\partial q} & \frac{\partial \mathbb{P}}{\partial p} \end{pmatrix}$$
$$p = p(Q, \mathbb{P})$$

if M satisfies $\boxed{M\begin{pmatrix} & 1 \\ -1 & \end{pmatrix} M^T = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}}$ $\leftarrow$ symplectic transform

then the form of the Hamilton Eq is invariant, ie

$$\boxed{\begin{aligned}\dot{q} &= \frac{\partial H}{\partial p} \\ \dot{p} &= -\frac{\partial H}{\partial \dot{q}}\end{aligned}} \xrightarrow{H(q,p)\to H(Q,\mathbb{P})} \boxed{\begin{aligned}\dot{Q} &= \frac{\partial H}{\partial \mathbb{P}} \\ \dot{\mathbb{P}} &= -\frac{\partial H}{\partial \dot{Q}}\end{aligned}}$$

⊛ Matrix groups

- $\boxed{n \times n \text{ unitary matrix} \quad SU(n):}$ $\quad \mathcal{U}^\dagger \mathcal{U} = 1$ with $\det \mathcal{U} = 1$

  a vector in d-dimensional complex linear space

  $$\psi = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}, \quad \phi = \begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix} \qquad \psi'^\dagger \phi' = \psi^\dagger \phi$$

  $$\psi' = \mathcal{U}\psi, \quad \phi' = \mathcal{U}\phi \Rightarrow \boxed{\sum_{a=1}^{n} c_a^* d_a = \sum_{a=1}^{n} c_a'^* d_a'}$$

- $\boxed{n \times n \text{ orthogonal matrix} \quad SO(n)}$ $\quad O^T O = 1$ with $\det O = 1$

  $$\left.\begin{matrix} \psi' = O\psi \\ \phi' = O\phi \end{matrix}\right\} \Rightarrow \boxed{\begin{matrix} \psi'^T \phi' = \psi \phi' \\ \sum_{a=1}^{n} c_a d_a = \sum_{a=1}^{n} c_a' d_a' \end{matrix}}$$

- $\boxed{2n \times 2n \text{ symplectic matrix}}$ $\quad M^T \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} M = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$

  $$\psi = \begin{pmatrix} x_1 \\ \vdots \\ x_n, \\ y_1 \\ \vdots \\ y_n \end{pmatrix}, \quad \phi = \begin{pmatrix} u_1 \\ \vdots \\ u_n, \\ w_1 \\ \vdots \\ w_n \end{pmatrix}$$

  $$\psi^T \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \phi = x^T w - y^T u = \sum_{a=1}^{n} (x_a w_a - y_a u_a)$$

  is invariant under symplectic transform.

☆ Applications in QM — Wigner, Weyl

- Atomic, moleculer spectra.
- theory of angular momentum
- wavefunction of identical particles

☆ Application in candensed matter

- crystal and band structure / Bloch theory
- phase transition, criticality
- Majorana fermion / topological theory

☆ High energy

- isospin
- unification: $\begin{cases} SU(3) \text{ — QCD} \\ SU_L(2) \otimes U(1) \text{ — EM \& weak interaction} \\ \text{grand unification (?) } SU(5) \\ \text{string theory} \end{cases}$
- Yang- Mills theory