Group theory — HW I

Problem 1: ① Prove that there are only five convex regular polyhedra. They have been known since 360 BC, and are called Platonic solids. (Plato associtated tetrahedron → fire, cube → earth, octahedron → air, icosahedron → water, dodecahedron → "ether" by Aristotle).

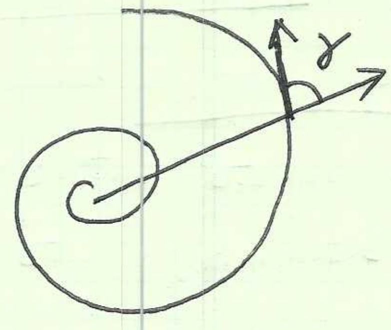
② In a dual pair of polyhedra, the vertices of one polyhedron correspond to the faces of the other. Show that tetrahedron is self-dual, cube and octahedron are a dual pair, and icosahedron and dodecahedron are also a dual pair. We will use these results later in this class.

Hint: use Schläfl symbol (n, m) to denote a regular polyhedron. "n" is the numbe of edges of each face, and "m" is the number of faces meeting at a vertex. Use the Euler character $V-E+F=2$ to derive an equality and seek integer number solution.

You can look in wiki on items "regular polyhedron", "Platonic solids", "Euler characteristic", "Kepler's platonic solids of the mode Solar system".

Prob 2: ① Prove that the logarithmic spiral $\rho = a e^{b\theta}$ has
a combined operations of rotation and scaling.
symmetry of

② Prove that at each point of the log-spiral, the tangent line and the radial line form a fixed angle.



a nautilus shell

③ Can you search for more interesting properties of log-spiral and tell me? This curve is on the tomb stone of Jocab Bernoulli "Eadem mutata resurgo".

Prob 3: We use the permutation symmetry to solve the cubic equation. Without loss of generality, we consider

$x^3 + px + q = 0$, for which the three roots x_1, x_2, x_3 satisfy

$x_1 + x_2 + x_3 = 0$. Certainly, according to Viéte theorem, there're two other symmetric polynominals $\begin{cases} x_1x_2 + x_1x_3 + x_2x_3 = p \\ x_1 x_2 x_3 = -q \end{cases}$. But they are non-linear, and difficult to use.

(a) We conside the S_3 group and use x_1, x_2, x_3 to form representations of S_3. Certainly, $x_1 + x_2 + x_3$ is one and it is invariant, or, form the A_1 representation – defined later. Then we form the other two

$$\begin{cases} x_1 + \omega x_2 + \omega^2 x_3 = A, \\ x_1 + \omega^2 x_3 + \omega x_2 = B \end{cases} \quad \text{where } (\omega = e^{i\frac{2}{3}\pi})$$

certainly, they are no longer invariant under S_3 permutations.

Take A and B as independent basis. Figure out how A and B transform under S_3: $\begin{pmatrix} 1&2&3 \\ 1&2&3 \end{pmatrix}, \begin{pmatrix} 1&2&3 \\ 2&3&1 \end{pmatrix}, \begin{pmatrix} 1&2&3 \\ 3&1&2 \end{pmatrix}, \begin{pmatrix} 1&2&3 \\ 2&1&3 \end{pmatrix}, \begin{pmatrix} 1&2&3 \\ 3&2&1 \end{pmatrix}, \begin{pmatrix} 1&2&3 \\ 1&3&2 \end{pmatrix}$.

Represent each permutation as a 2×2 matrix following the procedure blow. Say, apply $\begin{pmatrix} 1&2&3 \\ 2&1&3 \end{pmatrix}$, then

$$A \to x_2 + \omega x_1 + \omega^2 x_3 = aA + bB$$

$$B \to x_2 + \omega^2 x_3 + \omega x_3 = cA + dB$$

where a, b, c, d are constants you need to figure out, then

$\begin{pmatrix} 1&2&3 \\ 2&1&3 \end{pmatrix}$ maps to $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Each permutation corresponds to a 2×2 matrix, and these matrices form the E-represestation of S_3.

Write down the matrices of all the 6-elements of S_3 in the basis of A and B.

(b) Show that A^3+B^3 and AB are invariant under S_3, and they can be represented by the symmetric polynominals of the Vieta theorem.

(c) Use A^3+B^3 and A^3B^3, which are polynomials of p and q, solve A and B. Then combine
$$\begin{cases} x_1+x_2+x_3=0 \\ x_1+\omega x_2+\omega^2 x_3=A \\ x_1+\omega^2 x_3+\omega x_2=B \end{cases},$$
we arrive at the Cardano formula
$$\begin{cases} x_1=u+v \\ x_2=u\omega+v\omega^2 \\ x_3=u\omega^2+v\omega \end{cases}$$
where $u_1=\left(-\frac{q}{2}+\sqrt{\Delta}\right)^{1/3}$

$v=\left(-\frac{q}{2}-\sqrt{\Delta}\right)^{1/3}$

and $\Delta=\left(\frac{q}{2}\right)^2+\left(\frac{p}{3}\right)^3$.

(d) History of complex unit "i". Convince your self, for a real coefficient equation: $p, q \in \mathbb{R}$. If x_1, x_2 and x_3 are all real, then $\Delta<0$. We have to meet $\sqrt{-1}$ in the intermediate steps to reach real roots. (rigorous proof is not required).

Prob 4: Consider a 3D isotropic harmonic oscilator. Define the ~~transform~~ for its phonon anihilation operators $\vec{a} = \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} \to \begin{pmatrix} a_x' \\ a_y' \\ a_z' \end{pmatrix} = \mathcal{U} \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix}$, where

$\mathcal{U}$ is a 3×3 unitary matrix.

(a) Show that the Hamitonian is invariant under this transformation.

(b) $\mathcal{U}(3)$ group can be decomposed to a $\mathcal{U}(1) \otimes SU(3)$. The $\mathcal{U}(1)$ part is a pure phase. It's generator is just the identity matrix $I = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$. The corresponding conserved quantity is $a^{+} \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} a$

where $a^{+} = (a_x^{+} \; a_y^{+} \; a_z^{+})$. what's this quantity?

(c) The other part $SU(3)$ has 8 generators — Gellmann matrix

σ_1-like $\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ $\lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

σ_2-like $\lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}$ $\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$

σ_3-like $\lambda_3 = \begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix}$ $\underbrace{\sqrt{\tfrac{1}{3}}\begin{pmatrix} 1 & & \\ & 1 & \\ & & -2 \end{pmatrix}}_{\lambda_8}$

They give rise to eight conserverd quantities $O_i = a^{+} \lambda_i a$.

Show that for the σ_2-like, i.e. O_2, O_5, and O_7, they are familar quantities. What are they?

The other five i.e. σ_1 and σ_3 - like ones $O_{1,4,6}$ and $O_{3,8}$ are gudrapoles. Please write down their expressions.

(d) Write the degeneracy pattern of the 3D harmonic oscillators. and discuss how it is beyond the $2l+1$ fold degeneracy from angular momentum conservation. ~~you decompose them in~~