Real space RG — Kadanoff's picture (I)

§ Block spins

Instead of directly calculating partition function, Kadanoff developed an insight that coupling constant of an effective Hamiltonian should depende on the length scale over which the order parameter is defined. We will use the Ising model as an example:

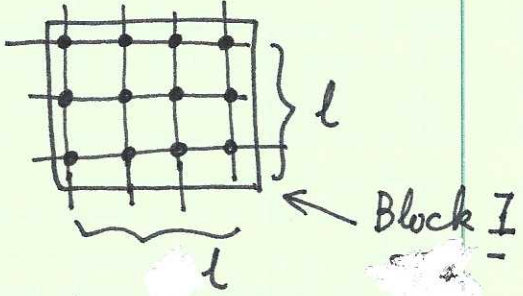
$$\beta H = -\beta J \sum_{\langle ij\rangle} S_i S_j - \beta B \sum_i S_i = -K \sum_{\langle ij\rangle} S_i S_j - h \sum_i S_i$$

where $K = \beta J$, $h = \beta B$ are dimensionless parameter.

Use $f_s(t, h)$ to denote the singular part of the free energy per spin near T_c. We define coarse ~~over~~ graining parameter l, such that

$$a \ll la \ll \xi(T).$$

A block with each site length l:



define block spin

$$S_I \equiv \frac{1}{|\bar{m}_l|} \frac{1}{l^d} \sum_{i \in I} S_i$$

(d is the space dimension)

and $\bar{m}_l \equiv \frac{1}{l^d} \sum_{i\in I} \langle S_i \rangle$, thus $\langle S_I \rangle = \pm 1$.

We should be able to express the effective Hamiltonian at the level of Block spin with size la, i.e. $\boxed{-\beta H_l = K_l \sum_{\langle IJ\rangle}^{Nl^{-d}} S_I S_J + h_l \sum_{I}^{Nl^{-d}} S_I}$, Then the correlation length measured in the bigger lattice constant ℓa is smaller i.e. $\xi = \xi_\ell(\ell a) = \xi_1 a$, thus $\xi_\ell = \xi_1/\ell$.

Since $\xi_\ell < \xi_1$, the effective Hamiltonian must be futher away from criticality than the original system, at a different reduce temperature t_ℓ.

Similarly the effective magnetic field h is also rescaled to h_ℓ.

$$h \sum_i S_i \simeq h\, \bar{m}_\ell\, \ell^d \sum_I S_I \equiv h_\ell \sum_I S_I \Rightarrow h_\ell = h\, \bar{m}_\ell\, \ell^d .$$

For the coarsed system, we have

$$N \ell^{-d} f_s(t_\ell, h_\ell) = N f_s(t, h) \Rightarrow \boxed{f_s(t_\ell, h_\ell) = \ell^d f(t, h)}$$

Block spin free energy.

Near the critical regim, we assume

$$\boxed{\begin{array}{ll} t_\ell = t\,\ell^{y_t}, & y_t > 0 \\ h_\ell = h\,\ell^{y_h}, & y_h > 0 \end{array}},$$

we arrive at the homogeneous form of free energy

$$f_s(t, h) = \ell^{-d} f_s(t\ell^{y_t}, t\ell^{y_h}).$$

if we set $\ell = |t|^{-1/y_t}$, we have

$$\boxed{\begin{array}{l} f_s(t.h) = |t|^{d/y_t} f(1, h|t|^{-y_h/y_t}) = |t|^{2-\alpha} F_f(h/|t|^{\Delta}) \\ \text{with } \Delta \equiv y_h/y_t, \quad 2-\alpha \equiv d/y_t \quad \text{and } F_f(x) \equiv f_s(1, x). \end{array}}$$

This was the starting point of the scaling hypothesis.

* Correlation functions

The correlation function of the block spin

$G(r_\ell, t_\ell) \equiv \langle S_I S_J \rangle - \langle S_I \rangle \langle S_J \rangle$. r_ℓ is the distance between the centers of the blocks I and J in the units of la, i.e $r_\ell (la) = ra$.

Then what's relation between $G(r_\ell, t_\ell)$ and $G(r,t)$?

The average magnitization $\bar{m}_\ell = \frac{h_\ell \ell^{-d}}{h} = \ell^{y_h - d}$.

The correlation function transform as

$$G(r_\ell, t_\ell) = \frac{1}{|\bar{m}_\ell|^2} \frac{1}{\ell^{2d}} \sum_{i \in I} \sum_{j \in J} \langle S_i S_j \rangle - \langle S_i \rangle \langle S_j \rangle$$

$$= \frac{1}{(\ell^{y_h - d})^2} \frac{1}{\ell^{2d}} \ell^d \ell^d \langle S_i S_j \rangle - \langle S_i S_j \rangle \simeq \ell^{2(d - y_h)} G(r,t)$$

add h, we have

$$G(r_\ell, t_\ell, h_\ell) = \ell^{2(d-y_h)} G(r,t,h)$$

or $$\boxed{G(r/\ell, t\ell^{y_t}, h\ell^{y_h}) = t^{2(d-y_h)} G(r,t,h)}$$

Set $l = t^{-1/y_t}$ $\Rightarrow$

$$G(r,t,h) = t^{2(d-y_h)/y_t} G(r t^{1/y_t}, h t^{-y_h/y_t}, 1)$$

since $r t^{1/y_t}$ is combined together, we may reorganize

$$G(r,t,h) = r^{-2(d-y_h)} (r t^{1/y_t})^{2(d-y_h)} G(r t^{1/y_t}, h t^{-y_h/y_t}, 1)$$

$$\triangleq r^{-2(d-y_h)} F_G(r t^{1/y_t}, h t^{-y_h/y_t})$$

This is just the scaling form that we assumed in Fisher scaling law: with the follwing correspondence

$$\boxed{\nu = 1/y_t, \quad 2(d-y_h) = d-2+\eta, \quad \Delta = y_h/y_t}$$

In Kadanoff's picture, we do not ~~treat~~ all the freedom at once. but progressively coarse graining the degree of freedom and coupling constants vary with the change of length scale.

Real space RG — 2d Ising model (II)

Now let us apply Kadanoff's idea to the 2D Ising model on triangular lattice (we don't use square lattice for technical reasons). We will follow the following steps

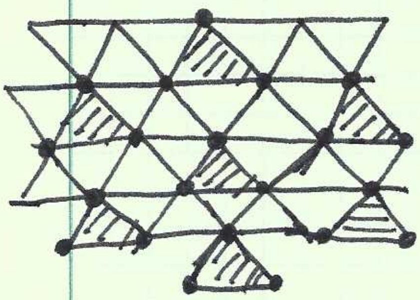
① Define block spin cluster and values for block spin,

② calculate partial trace,

③ find the RG transformation,

④ find the fixed points,

⑤ calculate the critical exponents.



$$H = -J \sum_{\langle ij \rangle} S_i S_j - \mu_B B \sum_i S_i$$

and $\mathcal{H} = -\beta H = \overset{\bar{\vee}}{} K \sum S_i S_j \;\; h \sum S_i$

with $K = J/k_B T$ and $h = \dfrac{\mu_B B}{k_B T}$.

The traditional method is to calculate

$$Z = \sum_{\{S_i\}} e^{-\mathcal{H}(\{S_i\}, K, h)}$$ and we obtain

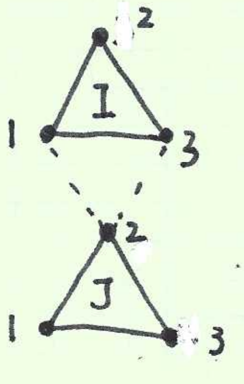
$F = -k_B T \ln Z = F(K, h)$, but Kadanoff did not aim at an exact solution of $Z(K, h)$.

① we define Kadanoff blocks as above: each block ansists of three sites. The center of clusters also form a triangular lattice with $a' = \sqrt{3}\, a$. The number of clusters $N' = N/3$.

$$\mathcal{H} = -K \sum_{\langle ij \rangle} S_i S_j - h \sum_i S_i = \mathcal{H}_0 + V$$

$\mathcal{H}_0$ is the intra-cluster term defineel as (we use I, J as block index)

$$\mathcal{H}_0 = \sum_I \mathcal{H}_0^I = \sum_I -K \left(S_1^I S_2^I + S_2^I S_3^I + S_3^I S_1^I \right)$$

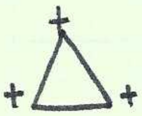


$$V = \sum_{\langle IJ \rangle} -K \left(S_1^I S_2^J + S_3^I S_2^J \right) + \cdots - h \sum_{i=1}^{N} S_i$$

we use the majority rule to define the value of block spin.

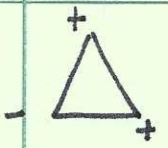
$$S_I = \mathrm{Sgn}\left(S_1^I + S_2^I + S_3^I \right)$$

spin anfing $(S_1^I, S_2^I, S_3^I) \longrightarrow (\overset{\text{blockspin}}{S_I} \qquad \overset{\text{internal index)}}{\sigma}$

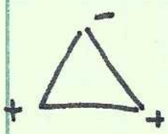


$(1 \quad 1 \quad 1) \longrightarrow (1 \qquad 1)$

$(1 \quad 1 \quad -1) \longrightarrow (1 \qquad 2)$



(1 -1 1) → (1 3)



(-1, 1 1) → (1 4)

Similarly, we also have the other four configurations

(S_I, σ): (-1 1), (-1, 2), (-1, 3) and (-1, 4).

The above rule define the mapping from $S_i^I \to S'_I$.

(' means block variable, or, slow variable).

② Partial trace

$$Z = \sum_{\{S_i\}} e^{-H\{S_i\}} = \sum_{\{S'_I\}} \left\{ \sum_{\{\sigma\}}{}' \, e^{-(H_0 + V)} \right\}$$

$$= \sum_{\{S'_I\}} e^{-H'\{S'_I\} - G}$$

$$e^{-H'\{S'_I\} - G} \equiv \sum_{\{\sigma_I\}}{}' \, e^{-(H_0+V)} = A \cdot \left\langle \Sigma' e^{-V} \right\rangle_0$$

where $A \equiv \sum_{\{\sigma_I\}}{}' \, e^{-H_0}$, and $\left\langle \Sigma' e^{-V} \right\rangle_0 \equiv \dfrac{\sum_{\{\sigma_I\}}{}' \, e^{-V} e^{-H_0}}{\sum_{\{\sigma_I\}}{}' \, e^{-H_0}}$

$\left\langle \Sigma' e^{-V} \right\rangle_0$ is doing averge in a fixed configure of block spin S'_I (slow variable) but average over the internal index σ_I. ↖(fast variable).

This is called partial trace.

$$\langle e^{-V}\rangle_0 = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\langle V^n\rangle_0$$

$$\Rightarrow Z = \sum_{\{S_I\}} e^{-(\mathcal{H}'+G)} \quad \text{and} \quad e^{-(\mathcal{H}'+G)} = A\langle e^{-V}\rangle_0$$

$$\Rightarrow \mathcal{H}' + G = -\ln A - \ln\langle e^{-V}\rangle_0$$

set $G = -\ln A$, which is an uninteresting part.

$$\ln\langle e^{-V}\rangle_0 = \ln\left(1 - \langle V\rangle_0 + \frac{1}{2!}\langle V^2\rangle_0 + \cdots\right)$$

$$= \left\{-\langle V\rangle_0 + \frac{1}{2!}\langle V^2\rangle_0 + \cdots\right\} - \frac{1}{2}\left\{-\langle V\rangle_0 + \frac{1}{2!}\langle V^2\rangle_0 + \cdots\right\}^2 + \frac{1}{3}\left\{-\langle V\rangle_0 + \cdots\right\}^3 + \cdots$$

$$= -\langle V\rangle_0 + \frac{1}{2!}\left\{\langle V^2\rangle_0 - \langle V\rangle_0^2\right\} - \frac{1}{3!}\left\{\langle V^3\rangle_0 - 3\langle V\rangle_0\langle V^2\rangle_0 + 2\langle V\rangle_0^3\right\} + \cdots$$

③ RG transformation:

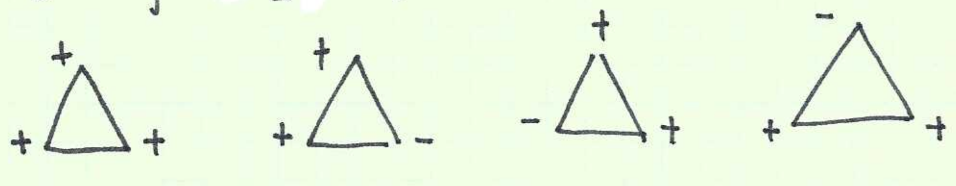
keep to the first order

$$\langle V\rangle_0 = \sum_{\langle IJ\rangle} -K\left(\langle S_1^I S_2^J\rangle_0 + \langle S_3^I S_2^J\rangle_0\right) - h\sum_{\langle I\rangle}\left(\langle S_1^I\rangle + \langle S_2^I\rangle + \langle S_3^I\rangle\right)$$

$$\boxed{\langle V\rangle_0 = -2K\sum_{\langle IJ\rangle}\langle S_1^I\rangle_0\langle S_2^J\rangle_0 - 3h\sum_{\langle I\rangle}\langle S_1^I\rangle_0}$$

For the case of $S_I = 1$,

I: block index



$\mathcal{H}_0^I$ $-3K$ K K K

$$\sum_{\sigma=1,2,3,4}' e^{-\mathcal{H}_0} = e^{3K} + 3e^{-K} \triangleq g_0$$

For the case of $S_I = -1$, we have the same $\sum_{\sigma=1,2,3,4}' e^{-\mathcal{H}_0} = e^{3K} + 3e^{-K}$

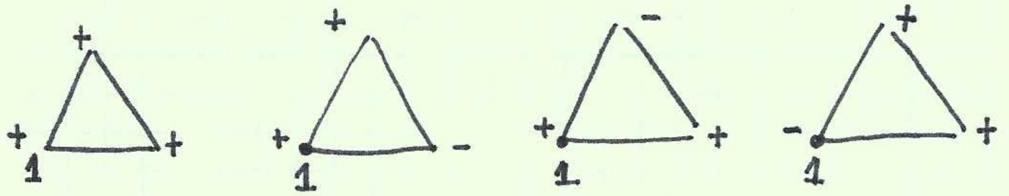
$$A = \sum_{\{\sigma_I\}}{}' e^{-\mathcal{H}_0} = \sum_{\{\sigma_I\}}{}' \prod_I e^{-H_0^I} = \prod_I \sum_{\substack{\sigma_I = 1,2,3\\ 4}}{}' e^{-H_0^I}$$

$$= \prod_1^{N'} g_0 = g_0^{N'} = (e^{3K} + 3e^{-K})^{N/3}$$

A does not depend on the distribution of $\{S_I'\}$.

$G = \ln A = N' \ln g_0$ — smooth part of free energy and we're not interested.

$$\langle S_1^I \rangle_0 \equiv \sum_{\{\sigma_I\}}{}' S_1^I e^{-\mathcal{H}_0} \Big/ \sum_{\{\sigma_I\}}{}' e^{-\mathcal{H}_0} = \frac{\sum'_{\sigma_I = 1\sim 4} S_1^I e^{-\mathcal{H}_0^I}}{\sum'_{\sigma_I = 1\sim 4} e^{-\mathcal{H}_0^I}}$$



$\mathcal{H}_0^I$	$-3K$	K	K	K
$e^{-\mathcal{H}_0^I}$	e^{3K}	e^{-K}	e^{-K}	e^{-K}
S_1^I	1	1	1	-1

$$\Rightarrow \text{If } S_I' = 1 \Rightarrow \langle S_1^I \rangle_0 = \frac{e^{3K} + e^{-K}}{e^{3K} + 3e^{-K}} .$$

Similarly, if $S_I' = -1$, we have $\langle S_1^I \rangle_0 = -\dfrac{(e^{3K} + e^{-K})}{e^{3K} + 3e^{-K}}$

$$\Rightarrow \langle V \rangle_0 = -2K \sum_{\langle IJ \rangle} \langle S_1^I \rangle_0 \langle S_2^I \rangle_0 - 3h \sum_I \langle S_1^I \rangle_0$$

$$\langle V\rangle_0 = -2K\left(\frac{e^{3K}+e^{-K}}{e^{3K}+3e^{-K}}\right)^2 \sum_{\langle IJ\rangle} S_I' S_J' - 3h\left(\frac{e^{3K}+e^{-K}}{e^{3K}+3e^{-K}}\right) \sum_I S_I'$$

$$\Rightarrow \boxed{\mathcal{H}'_{\{S_I\}} \simeq \langle V\rangle_0 = -K'\sum_{\langle IJ\rangle} S_I'S_J' - h'\sum_I S_I'}$$

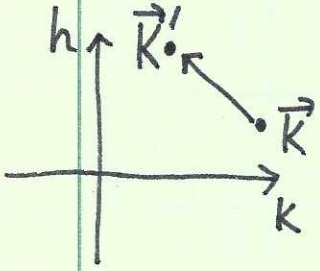
The new Hamiltonian has the same form as before.

$$\Rightarrow \begin{cases} K' = 2K\left(\dfrac{e^{3K}+e^{-K}}{e^{3K}+3e^{-K}}\right)^2 \\ h' = 3h\left(\dfrac{e^{3K}+e^{-K}}{e^{3K}+3e^{-K}}\right) \end{cases}$$

Recurisive relation

Define $\vec{K} \equiv (K, h)$, under the RG transformation

$$\mathcal{H}_{\{S_i\}} \xrightarrow{R(K)} H'_{\{S_I'\}} \xrightarrow{R(K)} \mathcal{H}''_{\{S_I''\}} \longrightarrow \cdots$$



④ Find the fixed point of R transformation

if $\vec{K}' = R(\vec{K})$, if $\vec{K}'^* = R(\vec{K}^*)$, then $\vec{K}^*$ is fixed point.

Unstable fixed point $\longleftrightarrow$ critical point

$a' \longrightarrow a' = l\, a$

$\xi' a' = \xi a \Longrightarrow \xi' = \frac{1}{l}\xi$

at CP point $\infty \longrightarrow \infty$

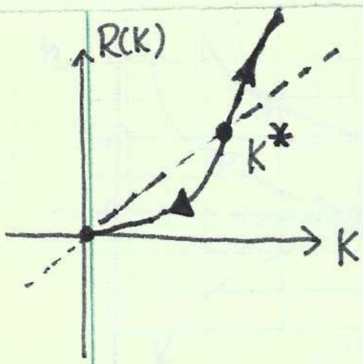
Solve p fixed point

$$\begin{cases} K^* = 2K^* \left(\dfrac{e^{3K^*}+e^{-K^*}}{e^{3K^*}+3e^{-K^*}}\right)^2 \\ h^* = 3h^* \left(\dfrac{e^{3K^*}+e^{-K^*}}{e^{3K^*}+3e^{-K^*}}\right) \end{cases}$$

$$\Rightarrow \begin{cases} K^* = 0,\ \infty \\ \dfrac{1}{\sqrt{2}} = \dfrac{e^{3K^*}+e^{-K^*}}{e^{3K^*}+3e^{-K^*}} = \dfrac{e^{4K^*}+1}{e^{4K^*}+3} \\ \to K^* = \tfrac{1}{4}\ln(1+2\sqrt{2}) = 0.34 \end{cases}$$

$$\Rightarrow h^* = 0,\ \infty$$

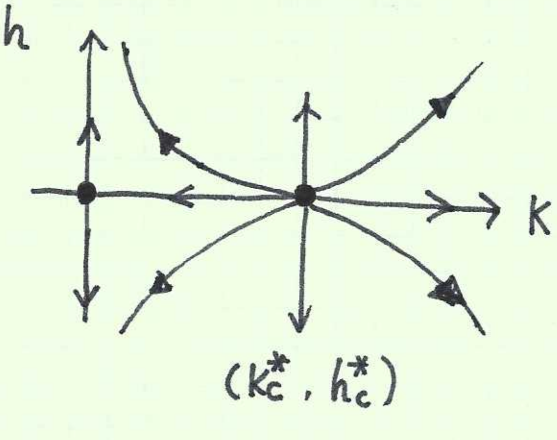
flow diagram



$$\begin{cases} K_c^* = 0.34 \text{ (not bad)} \\ h_c^* = 0 \end{cases}$$

The onsager solution on triagular lattice

$$K_c = \tfrac{1}{4}\ln 3 = 0.27$$



⑤ Calculation of critical exponents

recall widem scaling law: $\alpha\ \beta\ \gamma\ \delta\ \eta,\ \gamma$: p and q

$$f(\lambda^p t, \lambda^q h) = \lambda f(t,h) \quad \text{with } t = \frac{T-T_c}{T_c},\ h = \frac{\mu_B B}{k_B T}$$

or $f(t,h) = \frac{1}{\lambda} f(\lambda^p t, \lambda^q h)$

check in the Block spin system:

$$Z(K,h,N) = \sum_{\{S_i\}} e^{-\mathcal{H}(\{S_i\},K\,h,N)} = e^{-G} \sum_{\{S_I'\}} e^{-\mathcal{H}'(\{S_I\},K',h',N'\}}$$

$$F(K,h,N) = -k_BT \ln Z(K,h,N) = +k_BT\,G \underbrace{-k_BT \ln Z(K',h',N')}_{F'(K'h'N')}$$

if we neglect the regular part contribution from G, we have

$$\Rightarrow F(K,h,N) = F(K',h',N') \quad \text{or}$$

$$f(K,h) = \frac{1}{\ell^d} f(K',h')$$

set $\lambda = \ell^d$, in the vicinity of the critical point, we linearize the RG transformation

$$\begin{cases} \delta K = K - K^* \\ \delta h = h - h^* \end{cases} \Rightarrow \begin{pmatrix} \delta K' \\ \delta h' \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{\partial K'}{\partial K}\Big|_*, & \frac{\partial K'}{\partial h}\Big|_* \\ \frac{\partial h'}{\partial K}\Big|_*, & \frac{\partial h'}{\partial h}\Big|_* \end{pmatrix}}_{\mathbb{R}\text{-matrix}} \begin{pmatrix} \delta K \\ \delta h \end{pmatrix}$$

$$\text{and} \begin{cases} K' = K'(K,h) \\ h' = h(K,h) \end{cases}$$

After linearization

$$\begin{pmatrix} \delta K' \\ \delta h' \end{pmatrix} = \mathbb{R} \begin{pmatrix} \delta K \\ \delta h \end{pmatrix}.$$

We diagonalize $\mathbb{R}$ as $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ with eigenvector $\vec{e}_1, \vec{e}_2$

$$\begin{matrix} \delta K = \delta u_{K1}\vec{e}_1 + \delta u_{K2}\vec{e}_2 \\ \delta h = \delta u_{h1}\vec{e}_1 + \delta u_{h2}\vec{e}_2 \end{matrix} \longrightarrow \begin{cases} \delta K' = \lambda_1 \delta u_{K_1}\vec{e}_1 + \lambda_2 \delta u_{K_2}\vec{e}_2 \\ \delta h = \lambda_1 \delta u_{h_1}\vec{e}_1 + \lambda_2 \delta u_{h_2}\vec{e}_2 \end{cases}$$

if $\lambda > 1$, then $\delta u' = \lambda\, \delta u \rightarrow$ increasing

$\rightarrow$ relevant eigenvalue

if $\lambda < 1$. irrelevant eigenvalue.

For the case of

$$K' = 2K\left(\frac{e^{3K}+e^{-K}}{e^{3K}+3e^{-K}}\right)^2 \quad \Rightarrow \mathbb{R} = \begin{pmatrix} 1.62 & 0 \\ 0 & 2.12 \end{pmatrix}$$

$$h' = 3h\left(\frac{e^{3K}+e^{-K}}{e^{3K}+3e^{-K}}\right)$$

already diagonalized

$$\Rightarrow f(\delta u_1, \delta u_2) = \frac{1}{\ell^d} f(\lambda_1 \delta u_1, \lambda_2 \delta u_2)$$

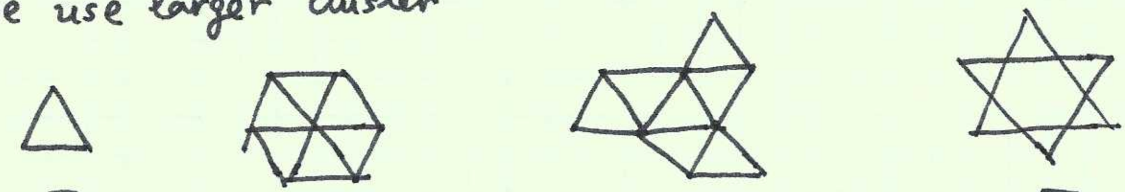
$$\Rightarrow \lambda = \ell^d, \quad \begin{cases} \lambda^p = \lambda_1 \\ \lambda^q = \lambda_2 \end{cases} \quad \Rightarrow \quad \ln\lambda = d\ln\ell$$

$$\begin{cases} p = \dfrac{\ln\lambda_1}{d\ln\ell} = \dfrac{\ln 1.62}{2\ln\sqrt{3}} = 0.44 \\[2ex] q = \dfrac{\ln\lambda_2}{d\ln\ell} = \dfrac{\ln 2.12}{2\ln\sqrt{3}} = 0.68 \end{cases}$$

exact value 2D Ising $p = 0.500$

$q = 0.938$

if we use larger cluster



ℓ:	$\sqrt{3}$	$\sqrt{7}$	3	$\sqrt{13}$
n:	3	7	9	13
	4	64	256	4096

$\uparrow$

dim: partial trace over internal states

n	$K_c = K^*$	p	q
3	0.34	0.442	0.685
7	0.30	0.459	0.740
9	0.31	0.485	0.730
13	0.30		
exact	0.275	0.500	0.938

★ if we keep high order term $\langle V^2\rangle_0 - \langle V\rangle^2$, $\langle V^3\rangle - 3\langle V^2\rangle\langle V\rangle + \langle V\rangle^3$

for $n=7$ cluster

$\Rightarrow K_c = 0.2752$, $p = 0.536$ $q = 0.938$.