Consider a Landau theory $\beta L = \int d^4 x \left( \frac{1}{2} \nabla \phi^2 + \alpha \phi^2 + \frac{\beta}{4} m^4 - \text{hm} \right)$

$Z = \int Dm \, e^{-\beta L}$.

Rescale: $\phi = \sqrt{\frac{\lambda}{2}} \, m$, ...., we arrive at the standard form

$Z = \int D\phi \, e^{-F(\phi)}$, with $F(\phi) = \int d^4 x \left( \frac{1}{2} (\nabla \phi)^2 + \frac{1}{2} \lambda_{0} \phi^2 + \frac{1}{4} \mu_0 \phi^4 \right)$.

1. Now we identify the dimensions of quantities of $\phi$ in $F(\phi)$.

$[F] = 1 \Rightarrow 2[\phi] \Rightarrow \frac{[\phi]}{[\phi]^2} = L^{-d}$

$[\lambda_{0}] [\phi]^2 = L^{-d}$

Similarly

$[\lambda_{0}] = L^{-d} (d_{\phi}^2 - 1)^2 = L^{-2}$

$[\mu_0] = L^{-d-4}$.

2. Define length scale $L = \xi(T) \equiv r_0^{-1/2}$

and dimensionless variables

$\varphi \equiv \frac{\phi}{\xi^{1-d/2}}$, $x \equiv \frac{r}{\xi}$, $\bar{\mu}_0 \equiv \frac{\mu_0}{\xi^{d-4}}$, $\xi \equiv r_0^{-1/2}$

then

$Z = \int D\varphi \, e^{-F(\varphi)}$ with

$F(\varphi) = \int d^4 x \frac{1}{2} (\nabla \varphi)^2 + \frac{1}{2} \varphi^2 + \bar{\mu}_0 \varphi^4$.

The last term is the interaction term. If $\bar{\mu}_0 \ll 1$, let us use perturbation theory.
\[ Z = \int D\phi \; e^{-F_0(\phi)} \; e^{-F_{\text{int}}}, \quad \text{where} \quad F_0(\phi) = \int d^d x \frac{1}{2} (\partial_x \phi)^2 + \frac{1}{2} \phi^2 \]

\[ F_{\text{int}} = \int d^d x \; \overline{\psi}_0 \psi^4 \]

\[ = \int D\phi \; e^{-F_0} \left( 1 - F_{\text{int}} + \frac{1}{2!} (F_{\text{int}})^2 + \cdots \right) \]

However, if \( d < 4 \), as \( t \to T_c \), or \( t \to 0 \), \( \delta \to \infty \), and \( \overline{\psi}_0 \to \infty \), thus perturbation theory fails.

If \( d > 4 \), as \( t \to 0 \), \( \overline{\psi}_0 \to 0 \), thus the perturbation theory becomes more and more accurate. \( \Rightarrow \) Upper critical dimension \( d = 4 \)!

We have another way to derive the criterion.

The critical region is entered at \( \overline{\psi}_0(t) \approx 1 \).

\[ \frac{\overline{\psi}_0}{\overline{\psi}_{\text{d.4}}} > 1 \]

\[ \frac{\overline{\phi}(t)}{\overline{\phi}(1)} = |t|^{-1} \]

\[ t \frac{1}{\delta^2} < \frac{\overline{\psi}_0}{\overline{\psi}_0^{\text{d.4}}} = \frac{\overline{\psi}_0^{\text{d.4}}}{\overline{\phi}^{\text{d.4}}} = \frac{\overline{\psi}_0 \delta^2}{\overline{\phi}^2} \frac{1}{\overline{\phi}^{\text{d.4}}} \]

\[ = \frac{\beta}{\alpha^2} \frac{1}{\overline{\phi}(1)} = \frac{1}{\Delta C \overline{\phi}(1)} \]

\( \Delta C \) is the mean field specific heat jump.

\[ \Delta C \]

The larger \( \overline{\phi}(1) \) and the larger \( \Delta C \), the more difficult to enter the critical region.
§ The origin of anomalous dimensions.

Let us look the two-point correlation function \( G(r) = \langle \phi(r) \phi(0) \rangle \).

The dimension of \( [\phi] = L^{1-d/2} \Rightarrow [G(r)] = L^{2-d} \). Its Fourier transform \( \tilde{G}(k) = \int d^dr \, G(r)e^{-ikr} \Rightarrow [\tilde{G}(k)] = L^2 \).

This is consistent with \( \tilde{G}(k) = \frac{1}{k^2 + r_0} \).

Check: Suppose that we rescale the length unit by a factor of \( \epsilon \): \( L' \equiv \epsilon L \).

Then

\[
G'(L'^2) = G(L^2) \Rightarrow G'(k', r_0') = G(k, r_0) \epsilon^{-2}
\]

\[
k'^{-1} = kL^{-1} \quad \text{with} \quad k' = \epsilon k
\]

\[
r_0'^{-1} = r_0 L^2 \quad \quad r_0' = \epsilon^2 r_0
\]

Q1: However, we defined before at \( T_c \), \( G(k, T_c) \sim k^{-2+\nu} \), which does not agree with above scaling! How can it be?

\( \text{at} \quad \nu \neq 0 \)

Q2: the correlation length \( \xi \sim r_0^{-1/2} \sim t^{-1/2} \) (because \( [\xi] = L \), \( [r_0] = L^{-2} \)).

How can it be: if we have \( \nu \neq 1/2 \).

Any critical exponents rather than mean-field one, means a violation of simple scaling dimension!
The reason is that we have also the microscopic length scale $a$, in addition to the correlation length scale $\xi$. Inside the following form

$$G(k, T, a) \sim a^2 k^{-2+\nu}$$

then $G'(k', T, a') = \xi^2 G(k, T, a)$. This can be termed as the $\phi$ has an anomalous dimension $\nu/2$.

Similarly, we should have

$$\xi = \xi_0^{-1/2} f(r_0 \alpha^2).$$

Near $t \to 0$, assume $f(x) \sim x^\theta$ as $x \to 0$, we have

$$\xi(t \to 0) \sim t^{-\nu/2+\theta} \alpha^{2\theta},$$

and thus $\nu = \frac{1}{2} - \theta$.\hfill\#