1. We have already mapped the 1D Ising model

\[ Z = \sum_{\{S\}} e^{\beta J \sum_i (S_i S_{i+1} - 1)} = \sum_{\sigma_1 \cdots \sigma_N} T_{\sigma_1 \sigma_2} \cdots T_{\sigma_N \sigma_1} = \text{tr} T^N \]

with \( T \) is a 2x2 matrix defined as \( e^{\hbar \omega_0 \sigma_i \cdot \text{const}} \)

\[ T = \begin{pmatrix} 1 & e^{-\beta J} \\ e^{-\beta J} & 1 \end{pmatrix} \Rightarrow Z = \text{tr} \left[ e^{N \hbar \omega_0 \sigma_i} \right] = \text{tr} \left[ e^{-\beta \hbar \omega_0} \right] \]

with \( \beta = \frac{N \omega_0}{2} \xrightarrow{N \to \infty} \infty \)

Relation \( \sinh 2\beta J \sinh \hbar \omega_0 = 1 \)

2. Let us consider the case of two-chain

\[ Z = \sum_{\{S\}} e^{\beta J \sum_{i=1}^{N} \sum_{j=1}^{2} S(i,j) S(i,j+1) + S(i,j) S(i+1,j)} \]

I change classic variable notation to \( S(i,j) \)

The transfer matrix is 4-dimensional

Define \( S_i = (S(i,1), S(i,2)) \), which takes 4 possible values \( S_i = (1,1), (-1,1), (1,-1), \) and \((-1,-1)\), Then

\[ Z = \sum_{S_1, \ldots, S_N} T_{S_1 S_2} T_{S_2 S_3} \cdots T_{S_N S_1} = \text{tr} T^N \]
\[ T_{S_1S_2} = e^{\beta J \sum_{j=1}^{2} S_{1j} S_{2j}} \cdot e^{\beta J S_{11} S_{12}} \]

\[ = \left[ T' \right]_{S_1S_2} \cdot \left[ T'' \right]_{S_2S_2} \]

\[ T' \text{ describes the two vertical bonds: independent evolution of two spins} \]

\[ S_1 \]
\[ (11) \]
\[ (11) \]
\[ \ldots \]
\[ (11) \]
\[ (-1-1) \]
\[ (-1-1) \]
\[ T' = (-1) \]
\[ (1-1) \]
\[ (1-1) \]
\[ \ldots \]
\[ (1-1) \]
\[ \ldots \]
\[ (1-1) \]

\[ = e^{\hbar \omega \frac{1}{2} \sum_{j=1}^{2} \sigma_{ij}(j)} \]

\[ T'' : \left[ \right] \text{ is diagonal: vertical bond} \]

\[ \Rightarrow T'' = \left[ e^{\beta J \sigma_{11} \sigma_{22}} \right]_{S_2S_2} \]
\[ T_{s_1s_2} = \left( e^{\frac{\mathcal{H}}{2}} \sum_{j=1}^{Z} \sigma_i(j) + \beta J \sigma_3(1) \sigma_3(2) \right)_{s_1s_2} \]

This picture can be generalized to \( M \)-chains, and \( T \) matrix represent the time-evolution of \( M \) spins, and thus \( T \) becomes \( 2^M \times 2^M \) dimensional. If we use periodical boundary condition along the \( j \)-direction, we have

\[ T = e^{\frac{\mathcal{H}}{2}} \sum_{j=1}^{M} \sigma_i(j) + \beta J \sum_{j=1}^{M} \sigma_3(j) \sigma_3(j+1) \]

\[ = e^{-\Delta Z H} \]

\[ \Rightarrow Z = tr\left(e^{-\frac{\mathcal{H}}{2}}\right) \]

where \( Z = N \sim 2^M \)

\[ \rightarrow \infty. \]

\[ H = -h \sum_{j=1}^{M} \sigma_i(j) - k \sum_{j=1}^{M} \sigma_3(j) \sigma_3(j+1) \]

\[ k/h = \frac{\beta J}{\mathcal{H}} \]

\[ \text{1D transverse field Ising model.} \]

And

\[ \text{we have mapped a 2D classic problem to 1D QM problem!} \]

\[ \text{2D classical phase transition} \rightarrow \text{1D Quantum phase transition.} \]
Now let's treat \( H = -K \sum_i (\gamma \sigma_i + \sigma_z(i) \sigma_z(i+1)) \) as a quantum model, and consider its ground state properties.

1. Strong coupling limit \( g \gg 1 \)

If \( g \to \infty \), the ground state is a paramagnetic state with each site spin parallel to \( \hat{x} \)-direction.

\[
|\psi_0\rangle = \prod_i |\uparrow_i\rangle_i, \quad \text{and} \quad \langle \psi_1 | \sigma_i^x \sigma_j^x | \psi_0 \rangle = \delta_{ij}.
\]

If \( g \) is large but finite, we expect \( \langle \psi_1 | \sigma_i^z \sigma_j^z | \psi_0 \rangle \sim e^{-|x_i - x_j|/g} \), i.e. short-range correlated. The excitation is to flip one site spin to \( \downarrow \), i.e.

\[
\Rightarrow \Rightarrow \cdots \Rightarrow \Rightarrow \Rightarrow \Rightarrow \Rightarrow |i\rangle = |\downarrow\rangle \prod_{i,j \neq i} \Rightarrow |\uparrow\rangle_j.
\]

All the states \( |i\rangle \) are degenerate at the limit \( g \to +\infty \). At \( 1/g \) level, the \( \sigma^z \sigma^z \) term couples different states together as

\[
\langle i | -K \sum_{n} \sigma_i^{z(n)} \sigma_{i+1}^{z(n+1)} | i \pm 1 \rangle = -K
\]

we can sum \( |1k\rangle = \frac{1}{\sqrt{N}} \sum_j e^{ikj} |1j\rangle \). Its eigen energy is

\[
\varepsilon_k = Kg \left[ 2 - \frac{3g}{2} \cos(k) + O(1/g^2) + \cdots \right]
\]
7 weak coupling \( g \ll 1 \)

two fold degeneracy \( |\uparrow\rangle \otimes |\uparrow\rangle, \ldots \) and \( |\uparrow\rangle \otimes |\downarrow\rangle \ldots \otimes |\downarrow\rangle \).

\( \sigma^z \) has long-range order. The low energy excited states are topological nature - kink.

\[ |\uparrow\rangle \otimes |\uparrow\rangle \cdots |\uparrow\rangle \otimes |\downarrow\rangle \otimes |\downarrow\rangle \cdots \]

\( i \quad i+1 \)

if we neglect the coupling between sections with different number of kinks, we can easily work out its energy dispersion

\[ E_k = k(2 - 2g\text{aw}k\alpha + \mathcal{O}(g^3)) \]

[ the \( \text{Kg} \) term builds up hoping of kinks ].

3 Exact solution of spectrum

Define non-local transformation. Jordan-Wigner transformation

\[
\begin{align*}
\sigma_i^z &= 1 - 2c_i^\dagger c_i \\
c_i^\dagger &= \prod_{j<i} (1 - 2c_j^\dagger c_j) c_i \\
c_i^- &= \prod_{j>i} (1 - 2c_j^\dagger c_j) c_i^\dagger \\
\end{align*}
\]

\[
\text{inverse: } c_i = (\prod_{j<i} \sigma_j^z) \sigma_i^+ \\
c_i^- = (\prod_{j<i} \sigma_j^z) \sigma_i^-
\]

Ex: please check that \( \{ c_i, c_j^\dagger \} = \delta_{ij} \), and thus \( c_i, c_i^\dagger \) are spinless fermion operators.

For transverse field Ising model, it's more convenient to do a further transform \( \sigma^z \to \sigma^x \) and \( \sigma^x \to -\sigma^z \).
\[
\sigma^X_i = 1 - 2c_i^+c_i \\
\sigma^z_i = -\prod_{j<i} (1-2c_j^+c_j)(c_i^+c_i)
\]

\[
H = -K\sum_i \left\{ g(1-2c_i^+c_i) + (c_i^+c_i^+) (c_{i+1}^+ + c_{i+1}^+) \right\}
\]

\[
= -K\sum_i \left( c_i^+c_{i+1} + c_{i+1}^+c_i + c_i^+c_{i+1}^+ + c_{i+1}^+c_i - 2g c_i^+c_i - g \right)
\]

\[
= K\sum_k \left( 2(g - \cos k) c_k^+c_k - 2i\sin k (c_k^+c_k^+ - c_k^+c_k) - g \right)
\]

\[
= K\sum_k \begin{bmatrix} c_k^+ & c_k \end{bmatrix} \begin{bmatrix} 2(g - \cos k) & 2i\sin k \\ 2i\sin k & -2(g - \cos k) \end{bmatrix} \begin{bmatrix} c_k^+ \\ c_k \end{bmatrix}
\]

\[
\Rightarrow \text{The excitation spectrum}
\]

\[
E_k = 2K ( (g - \cos k)^2 + \sin^2 k )^{1/2} = 2K ( 1 + g^2 - 2g\cos k )^{1/2}
\]

**Ex:** 1) Please diagonalize the above matrix by Bogoibolibin transform.

2) Check that \( E_k \) at \( g<<1 \) and \( g>>1 \), agrees with the approximate expression given above.

At both \( g>1 \), and \( g<1 \), because \( 1+g^2>2g \), the spectra of \( E_k \) is gapped. But at \( g=1 \), \( E_k = 4K\left|\sin \frac{k}{2}\right| \), the spectra is gapless, which indicate a quantum phase transition. Indeed, \( |g|<1 \) corresponds to topological pairing, and \( |g|>1 \) is topologically trivial pairing.
The pairing matrix \( \Delta_k = 2(\mathcal{G}_{\text{swk}}) \tau_1 - \text{Sinh} \tau_2 \) as \( k \) in the \( B_2 \), \( k \in [-\pi, \pi] \), if we represent \( \Delta_k \) as a 2-vector in the basis of \( \tau_1, \tau_2 \), we have

\[ g = 1 \]

If we come back to the spin language, we have an order/disorder transition.

**Duality (site-bond)**

\[
\begin{align*}
\mu_{n+1/2}^x &= \prod_{j=1}^{n} \sigma_j^x \\
\mu_{n+1/2}^z &= \sigma_n^z \sigma_{n+1}^z \\
\mu_{n+1/2}^x &= \sigma_n^z \sigma_{n+1}^z
\end{align*}
\]

\[
\begin{align*}
\sigma_n^2 &= \prod_{j=0}^{n-1} \mu_j^{x, y, z} \\
\sigma_n^x &= \mu_n^{x, y, z} \mu_{n+1/2}^{x, y, z}
\end{align*}
\]

in terms of \( \mu \Rightarrow \)

\[
H = -K \left[ g \sum_n \mu_{n-1/2}^{x, y, z} \mu_{n+1/2}^{x, y, z} + \mu_{n+1/2}^{x, y, z} \right]
\]

\[ g \rightarrow \frac{1}{g} \text{ self-duality} \]
What is $\mu$? the kink operator / disorder operator

$$|\Omega\rangle = T^{\dagger}_n |\uparrow\rangle_n \Rightarrow \mu^2_{n-1/2} |\text{vac}\rangle = |\downarrow\downarrow\cdots\downarrow\uparrow\uparrow\cdots\rangle_{n \to n+1/2}$$

Thus $g > 1$, $O_2$ disordered, $\leftrightarrow \mu$ ordered

$< 1 \Rightarrow$ ordered $\leftrightarrow \mu$ disordered

Furthermore back to 2D Ising model $\Rightarrow$ low $T < T_c \Rightarrow$ Wigner-Kramers duality

$\S$ Majorana Representation

$$\xi_1(n) = \frac{C^n + C_n}{\sqrt{2}} \quad \xi_2(n) = \frac{C^n - C_n}{-\sqrt{2} i} \Rightarrow \{\xi_i, \xi_j\} = \epsilon_{ij}$$

Ex: please verify that in the Majorana Rep

$$H = K \left( ig \xi_2(n) \xi_1(n) - i \xi_2(n) \xi_2(n+1) \right)$$

antimene version

$$\frac{H}{K} = -i \xi_2(n) (\xi_1(n+1) - \xi_1(n)) + i (g-1) \xi_2(n) \xi_1(n)$$

$$\Rightarrow \int dx \xi_2(-i\partial_x)\xi_1 - im\xi_1 \xi_2 \quad m = g-1$$

$$= \frac{1}{a}\int dx \xi^T \begin{pmatrix} \alpha \beta^T \pm \beta m \end{pmatrix} \xi, \quad \text{where} \quad \xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}, \quad \alpha = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \beta = \begin{pmatrix} \imath \\ -\imath \end{pmatrix}, \quad \beta = -i\partial_x$$