Existence of the PM-FM phase transition of the 2D Ising model

The kink argument in 1D is no longer valid in 2D, because the domain wall becomes a line whose size can reach the order of $L$. ($L$ is the length of the sample and $N = L^2$.) Naively, flip one spin cost the energy $2zJ$, where $z$ is the coordination number. When $T < \frac{2J}{z}$, every site has the probability at the order of 1 to be flipped, $\Rightarrow T_c$ is at order of $\frac{2J}{z}$. In fact, this is the mean-field $T_c$ given by the Bragg-Williams approximation.

We follow Peierls and Griffiths to show the existence of low $T$ ordered phase. In 2D, a general configuration can be represented by domain wall lines. We assume all spins on the surface are pinned to be spin up as an external field. The effects of the surface are expected to vanish as $L \to \infty$.

Configurations $\mapsto$ closed loops

The number of sites enclosed by this loop $\leq \left(\frac{b}{4}\right)^2$.

Denote $m(b)$ as the number of domain wall loops with length $b$

$$m(b) \leq 3^{b-1} N. \quad \text{The first bond can pick any position, as the next one has 3 directions to pick.}$$
Thus the number of spin down
\[ \langle N_{\downarrow} \rangle = \sum_{b} \left( \frac{b}{4} \right)^{2} \cdot N \cdot 3^{b-1} \cdot \text{Probability (loop of length b)}. \]

All configuration = (configuration with the loop b) + configurations without

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Thus its probility = \[ \frac{e^{-2\beta bJ}}{1 + e^{-2\beta bJ}} \]

where \( 2bJ \) is the energy cost

\[ < e^{-2\beta bJ} \]

\[ \Rightarrow \frac{N_{\downarrow}}{N} \leq \sum_{b=4,6,\ldots} \left( \frac{b}{4} \right)^{2} \cdot 3^{b-1} \cdot e^{-2\beta e b} = \frac{x^{2}}{3(1-x)} \left( 1 - \frac{3}{4}x + \frac{x^{2}}{4} \right) \]

with \( x = 9 e^{-2\beta e} \)

as \( \beta \) sufficiently large, or \( x \to 0 \), certainly we can have \( \frac{N_{\downarrow}}{N} < \frac{1}{2} \),

and thus we have ordered phase!

In fact, we can solve exactly \( T_c \) even without solve the partition function. — by Kramers–Wannier duality!
Kramers and Wannier's duality

In the absence of external field, the partition function of 2D Ising model of \( N = n \times n \) sites can be mapped to that of the same model but with a different temperature, i.e.

\[
\frac{\ln Z(N,\beta)}{N} = \frac{\ln Z(N,\beta^*)}{N} - \sinh 2\beta^*
\]

where \( \beta^* = -\frac{1}{2} \ln \tanh \beta \), or \( \sinh 2\beta \sinh 2\beta^* = 1 \). Thus the low temperature phase is mapped to the high temperature phase. If we assume that there are only two phases and one transition, we argue that the critical temperature \( \beta_c \) is the self-dual point.

\[
\frac{k_B T_c}{J} = \frac{2}{\ln(1+\sqrt{2})} = 2.269.
\]

In other words, the Ising model is self-dual, and we can solve \( T_c \) exactly even without solving the model! Next we will prove its self-duality.

\[
Z(N, \beta) = \sum_{\{\sigma_i\}} e^{\beta \sum_{\langle i,j \rangle} \sigma_i \sigma_j} = \sum_{\{\sigma_i\}} \prod_{\langle i,j \rangle} e^{\beta \sigma_i \sigma_j}
\]

\[
e^{\beta \sigma_i \sigma_j} = \cosh \beta + \sigma_i \sigma_j \sinh \beta
\]

\[
\Rightarrow \ Z(N, \beta) = \sum_{\{\sigma_i\}} \prod_{\langle i,j \rangle} \frac{1}{C_i C_j} C_k(\sigma_i \sigma_j)^{k_{ij}}
\]

with \( C = \cosh \beta \), \( C_1 = \sinh \beta \).
each $<ij>$ is assigned an integer $k_{ij} = 0 \text{ or } 1$ $<ij>$ is a bond

$$Z[N, \beta] = \sum_{\{\sigma_i\}} \sum_{\{k_{ij}\}} (C_{k_{ij}}, C_{k_{ij}} , \ldots) \prod_i \sigma_i^{n_i}$$

$$= \sum_{\{k_{ij}\}} \sum_n (C_{k_{ij}}, C_{k_{ij}} , \ldots) \prod_i (\sigma_i^{n_1}, \sigma_i^{n_2}, \ldots)$$

where $n_i = \sum_{<ij>} k_{ij}$

all the neighbors

$$\Rightarrow Z[N, \beta] = \sum_{\{k_{ij}\}} (C_{k_{ij}}, C_{k_{ij}} , \ldots) \prod_i \left(\sum_{\sigma_i} \sigma_i^{n_i}\right)$$

$$= \sum_{\{k_{ij}\}} C_{k_{ij}}, C_{k_{ij}} , \prod_i \left[1 + (-)^{n_i}\right]$$

$$\Rightarrow$$ only the configuration that every site $n_i = \sum_{<ij>} k_{ij} =$ even can survive. This is the constraint

$$\Rightarrow Z[N, \beta] = 2^N \sum_{\{k_{ij}\}}' (C_{k_{ij}}, C_{k_{ij}} , \ldots)$$

where $\sum'$ means the constraint $\sum_{<ij>} k_{ij} = 0 \text{ (mod } 2\text{)}$

for site $i$. 
Around site $i$ in the original lattice, there are four sites in the dual lattice. We assign dual spin variable $\mu_i$ (Ising) and we can solve the constraint as:

$$k_b = \frac{1}{2} (1 - \mu_i \mu_j)$$

the bond $b$ intersect with the bond $<i'j'>$ in the dual lattice.

$$\Rightarrow \sum k_b = \frac{1}{2} (4 - \mu_1 \mu_2 - \mu_2 \mu_3 - \mu_3 \mu_4 - \mu_4 \mu_1)$$

around $i$

because $\mu_1 \mu_2 + \mu_2 \mu_3 + \mu_3 \mu_4 + \mu_4 \mu_1 = (\mu_1 + \mu_3)(\mu_2 + \mu_4) = 2 = 0$ or $\pm 4$

$$\Rightarrow \sum k_b \equiv 0 \pmod{2}.$$ around $i$

"One question is that: for every allowed $k_{<ij>}$ configuration, does it always corresponds to a solution of $\mu_i$?" Apparently, flipping the sign of all $\mu_i$, does not change $k_b$, and thus there's 2 to 1 mapping.

$$\Rightarrow \quad Z = 2^{N/2} \sum_{\{\mu\}} C_{k_{<ij>}, k_{<i'j'>}}$$

each bond $<ij>$ intersect with a bond in $<i'j'>$ in the dual lattice. we denote $C_{k_{<ij>}}$ as $C_{k_{<i'j'>}}$. And we need to express $C_{k_{<ij>}}$ through $\mu_i$. 
\[ C_{k_{ij}} = k \sinh \beta + (1-k) \cosh \beta = \frac{1}{2} (1 - \mu_i \mu_j') \sinh \beta + \frac{1}{2} (1 + \mu_i \mu_j') \cosh \beta \]

\[ = \frac{1}{2} e^\beta (1 + \mu_i \mu_j' e^{-2\beta}) = \frac{e^\beta}{2 \cosh \beta'} [\cosh \beta' + \mu_i \mu_j' \sinh \beta'] \]

with \( \tanh \beta' = e^{-\beta} \)

\[ \Rightarrow C_{k_{ij}} = [2 \sinh 2\beta']^{-1/2} e^{\beta' \mu_i \mu_j'} \]

\[ \Rightarrow Z = \frac{1}{a} [\sinh 2\beta']^{-N} \sum_{\{\mu_j\}} \exp(\beta' \sum \mu_i \mu_j') \]

**Relation**

\[ \tanh \beta' = e^{-\beta} \]

\[ \sinh 2\beta \sinh 2\beta' = 1 \]