# Non-linear $\sigma$-model — lower critical dimension

Let us conside the Landau free energy for the n-componet case

$$F = \frac{1}{2}\sum_{i=1}^{N}(\vec{\nabla}\phi_i)^2 + \frac{\alpha}{2}\sum_{i=1}^{N}\phi_i^2 + \frac{\beta}{4}\left(\sum_{i=1}^{N}\phi_i^2\right)^2 .$$

say $n=3 \to$ Heisenberg magnet.

In the low temperature case $\alpha<0$, we have

$$\sqrt{\sum_{i=1}^{N}\phi_i^2} = \sqrt{\frac{|\alpha|}{\beta}} = |\bar{\phi}|$$

As we explained before, the fluctuations of the magnitude of $\phi$ field is suppressed : $\chi_{\perp}(k) = \frac{1}{2|\alpha| + k^2}$ , but its transverse fluctuation, or, direction of $\phi$ is large and powerlaw

$$\chi_{\parallel} = \frac{1}{k^2} \quad \text{— Gold-stone theorem.}$$

Now the question appears : we have already known that below $d_u = 4$, fluctuations can generate the non-trival Wilson-Fisher fixed point at $d=3$, but nevertheless it still corresponds to a transition to long-range ordered phase. But if we have a continous symmetry, such as the $O(n)$ symmetry, if $d$ is low, shall we be able to have long-range ordering at all? Even $\vec{\phi}$ can develop an non-zero amplitude, but the transverse fluctuations could disorder the configuration. We will show that there exist a lower critical dimension $d_c' = 2$ for continous symmetry. Thermal fluctuations forbid long-range ordering at any $T>0$. at $d \le d_c = 2$.

An evidence is the divergence of transverse fluctuations. Let us consider the $n=3$ case for example. Suppose we have a long range order along x-direction then $\chi_{\parallel}(k) = 1/k^2$. Let's ask the fluctuations along y-direction:

$$\boxed{\langle \phi_y^2(x) \rangle \sim \int_{1/L}^{\Lambda} \frac{d^d k}{(2\pi)^d} \frac{1}{k^2} \sim \int_{\frac{1}{L}}^{\Lambda} k^{d-3} dk} = \begin{cases} \frac{1}{d-2}\left[\Lambda^{d-2} - \left(\frac{1}{L}\right)^{d-2}\right] & (d \ge 3) \\ \ln \Lambda L & (d=2) \\ L - 1/\Lambda & (d=1) \end{cases}$$

at $d \ge 3$, the integral is convergent in the infra-red direction. But the transverse fluctuations diverge as $L \to \infty$ in 2D (logrithmically) and 1D (linearly). This means that our assumption of long-range ordering is not self-consistent.

Actually, the above result can be exactly proved known as Mermin-Wagner theorem. It have a quantum mechanical version that at $T=0$, the 1d system with continous symmetry cannot have long-range ordering. (There is a subtle case of ferro-magnetism in which there's no quantum fluctuation). But the ground states of 2D quantum system can possess long-range ordering.

(Quantum d-dimensional $\longrightarrow$ classical $d+1$-dimensional systems)

Now, we will also confirm through RG. Suppose that $T \ll T_{MF}$, i.e $\alpha < 0$ and large, such that we can freeze the amplitude fluctuation. For the case of $O(n)$ symmetry, we use the symbol $\vec{n}$ whose magnitude is already normalized. We write down the partition function as

$$\boxed{Z = \int Dn \; e^{-\int d^dx \frac{1}{2g}(\partial_\mu \vec{n})^2}}$$

with the constraint $|\vec{n}|^2 = 1$.

the unit of $g$ is $a_0^{d-2}$ where $a_0$ is the microscopic cut off, we define $g = u a_0^{d-2}$, and $u$ is dimensionless. This model is not free because of the constraint $|\vec{n}|^2 = 1$. If $g \to 0$, the fluctuation is suppressed, which is called the weak coupling limit. If $g \to \infty$, fluctuation is enhanced, which is called the strong coupling limit. The RG flow shows that at $d \leq 2$, there're only strong coupling fixed point ($g \to \infty$), which means that no-long range order can exist.

Now we do RG transformation: separate $\hat{n}$ into slow and fast degrees: (local frame are slow modes, $\phi^a$ are fast modes)

$$\hat{n}(x) = n^0(x)\sqrt{1 - \vec{\phi}^2(x)} + \sum_{a=1}^{n-1} \phi^a \hat{e}^a(x)$$

where $(\overset{\circ}{n}(x), \hat{e}^a(x))$ are local frame. define $\hat{e}^0(x) \equiv n^0(x)$ then $\hat{e}^\alpha(x) \cdot \hat{e}^\beta(x) = \delta_{\alpha\beta}$, $\alpha, \beta = 0, 1, \cdots n-1$.

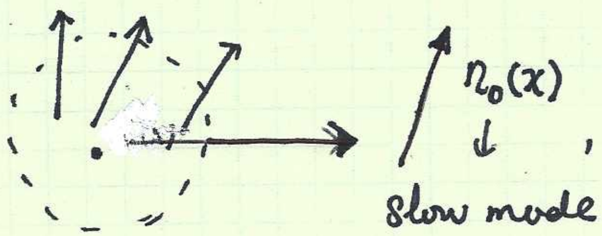
$$\partial_\mu \hat{n}^0 = \sum_a (\hat{e}^a \cdot \partial_\mu \hat{n}^0)\, \hat{e}^a = \sum_\alpha \tilde{A}_\mu^{a0} \hat{e}^a$$

define $\tilde{A}_\mu^{\alpha\beta} \equiv \hat{e}^\alpha \cdot \partial_\mu \hat{e}^\beta$, and thus $\tilde{A}_\mu^{\alpha\beta} = -\tilde{A}_\mu^{\beta\alpha}$

similarly $\partial_\mu \hat{e}^\alpha = \sum_b \tilde{A}_\mu^{ba} \hat{e}^b - \tilde{A}_\mu^{a0} \hat{n}^0$

we also have $\boxed{\sum_a (\tilde{A}_\mu^{a0})^2 = \sum_a (\hat{e}^a \cdot \partial_\mu \hat{n}^0)^2 = (\partial_\mu \hat{n}^0)^2}$

we determine an intermediate momentum scale $\hat{\Lambda} = \Lambda/\ell$,




$\phi^a(x)$ are transverse fluctuations: fast modes

$$\partial_\mu \hat{n} = n^0(x)\, \partial_\mu \sqrt{1-\Phi^2(x)} + \sqrt{1-\Phi^2(x)} \sum_a \tilde{A}_\mu^{a0} \hat{e}^a$$

$$+ \sum_a \phi^a \Big( \sum_b \tilde{A}_\mu^{ba} \hat{e}^b - \tilde{A}_\mu^{a0} \hat{n}^0 \Big) + \sum_a \partial_\mu \phi^a \hat{e}^a$$

$\tilde{A}^{ab} = -\tilde{A}^{ba}$

$$= n^0(x) \Big[ \partial_\mu \sqrt{1-\Phi^2(x)} - \sum_a \phi^a A_\mu^{a0} \Big) + \sum_a \Big[ \tilde{A}_\mu^{a0} \sqrt{1-\Phi^2(x)} + \sum_b \tilde{A}_\mu^{ab} \phi^b \Big]$$
$$+ \partial_\mu \phi^a \Big] \hat{e}^a$$

$$F = \frac{1}{2u a_0^{d-2}} \int d^d x \Big( \partial_\mu \sqrt{1-\Phi^2(x)} - \sum_a \phi^a A_\mu^{a0} \Big)^2$$
$$+ \sum_a \Big( \partial_\mu \phi^a + \sum_b \phi^b \tilde{A}_\mu^{ab} + \tilde{A}_\mu^{a0} \sqrt{1-\Phi^2(x)} \Big)^2$$

expand to $\phi$'s second order: $\partial_\mu \sqrt{1-\phi^2} = \partial_\mu (1 - \frac{1}{2}\phi^2) = \phi_a \partial_\mu \phi_a$

thus

$$\left(\underbrace{\partial_\mu\sqrt{1-\bar\phi^2}}_{\text{neglected}} - \sum_a \phi^a A^{a0}_\mu\right)^2 + \sum_a\left(\partial_\mu\phi^a + \sum_b \phi^b \tilde{A}^{ab}_\mu + \tilde{A}_\mu^{a0}\sqrt{1-\bar\phi^2(x)}\right)^2$$

$$\simeq \underbrace{\left(\sum_a \phi^a A^{a0}_\mu\right)^2 + \sum_a\left(\partial_\mu\phi^a + \sum_b \phi^b \tilde{A}^{ab}_\mu\right)^2}_{①} + \underbrace{\sum_a \tilde{A}^{a0\,2}_\mu \left(1-\bar\phi^2(x)\right)}_{②}$$

$$+ \underbrace{2\sum_a \partial_\mu\phi^a \tilde{A}^{a0}_\mu + 2\sum_{ab} \phi^b \tilde{A}^{ab}_\mu \tilde{A}^{a0}_\mu}_{\text{odd in } \phi \text{, average to zero with respect to fast modes}}$$

$$① + ② = \sum_a\left(\partial_\mu\phi^a + \sum_b \phi^b \tilde{A}^{ab}_\mu\right)^2 + \sum_{ba}\left[\phi^a\phi^b - \delta_{ab}\bar\phi^2(x)\right]\tilde{A}^{a0}\tilde{A}^{b0}$$

$$+ \sum_a\left(\tilde{A}^{a0}_\mu\right)^2$$

and $\sum_a \left(\tilde{A}^{a0}\right)^2 = \left(\partial_\mu \hat{n}^0\right)^2$ ← the slow variable

$$\Rightarrow Z \simeq \int_{\tilde\Lambda} D\hat{n}^0 \exp\left(-\int_{\tilde\Lambda} d^d x \; F\left(\partial_\mu \hat{n}^0\right)^2\right)$$

$$\cdot \int_\Lambda D\phi \exp\left[-\int_\Lambda dx^d \; F^{(2)}(\hat{n}; \phi)\right]$$

where $F^{(2)} = \frac{1}{2ua_0^{d-2}} \sum_a \left(\underbrace{\partial_\mu\phi^a + \sum_b \phi^b \tilde{A}^{ab}_\mu}_{\text{gauge covariant}}\right)^2 + \sum_{ab}\left(\phi^a\phi^b - \delta_{ab}\bar\phi^2(x)\right)\tilde{A}^{a0}\tilde{A}^{b0}$

let us expand $F^{(2)}$

① free part $\sum_a \left(\partial_\mu\phi^a\right)^2 \cdot \frac{1}{2u a_0^{d-2}}$

② interaction with the gauge fields

$2\partial_\mu \phi^a \phi^b \tilde{A}_\mu^{ab} \longrightarrow \langle \partial_\mu \phi^a \phi^b \rangle = 0$ over fast field

$(\tilde{A}_\mu^{ab})^2 (\phi^b)^2 \longrightarrow$ only contains field of $\hat{e}^a$ and $\hat{e}^b$ not in the original model

$$\tilde{A}_\mu^{a0} \tilde{A}_\mu^{b0} (\phi^a\phi^b - \delta_{ab}\phi^2) \xrightarrow[a=b]{\text{average}} (\tilde{A}^{a0})^2 (\phi^{2a} - \bar{\phi}^2)$$

we only need to average to the last term

$$\left\langle e^{-(\tilde{A}_\mu^{a0})^2 \frac{1}{2ua_0^{d-2}} (\phi^{2,a} - \bar{\phi}^2)} \right\rangle = e^{-(\tilde{A}_\mu^{a0})^2 \frac{1}{2ua_0^{d-2}} \langle \phi^{2a} - \bar{\phi}^2 \rangle}$$

$$\frac{1}{2ua_0^{d-2}} \langle \phi^{a,2} - \bar{\phi}^2 \rangle = \frac{\langle \phi^{a,2} \rangle (1-(n-1))}{2ua_0^{d-2}} = (2-n) \int_{\tilde{\Lambda}<k<\Lambda} \frac{dk}{k^2}$$

please notice the free field is $\sum_a (\partial_\mu \phi^a)^2 \frac{1}{2ua_0^{d-2}}$

$$(2-n) \int_{\tilde{\Lambda}<k<\Lambda} \frac{dk}{k^2} = (2-n) K_d \Lambda^{d-2} \begin{cases} (\tilde{\Lambda}/\Lambda)^{-1} - 1 & d=1 \\ -\ln \tilde{\Lambda}/\Lambda & d=2 \\ \frac{1}{d-2}\left[1 - \left(\frac{\tilde{\Lambda}}{\Lambda}\right)^{d-2}\right] & d=3 \end{cases}$$

$\Lambda \sim 1/a_0$

$$\equiv \frac{(2-n)}{a_0^{d-2}} K_d \Delta_d$$

In the above calculation, we have assumed the $O(n-1)$ symmetry where do average over fast mode. Since $\sum_a (\tilde{A}_\mu^{a,0})^2 = (\partial_\mu n^0(x))^2$,

we sum ~~together~~:

$$F' = \int_{\tilde{\Lambda}} \left( \frac{1}{2u_0 a_0^{d-2}} - \frac{(n-2) K_d \Delta_d}{a_0^{d-2}} \right) (\partial_\mu n^0)^2 \, d^d x$$

restore $\tilde{\Lambda} \to \Lambda$, $F' = \int_\Lambda d^dx' \left(\Lambda/\tilde{\Lambda}\right)^{d-2} \left( \frac{1}{2u_0 a_0^{d-2}} - \frac{(n-2)C\Delta d}{a_0^{d-2}} \right) (\partial'_\mu n^0)^2$

$x \to x' = \frac{\Lambda}{\tilde{\Lambda}} x$

$$= \int_\Lambda d^dx' \frac{1}{2a_0^{d-2}} \left[ \frac{1}{u_0} - (n-2) K_d \Delta d \right] \left(\Lambda/\tilde{\Lambda}\right)^{d-2} (\partial'_\mu n^0)^2$$

$$\Rightarrow \frac{1}{u} = \left[ \frac{1}{u_0} - (n-2) K_d \Delta d \right] \left(\Lambda/\tilde{\Lambda}\right)^{d-2}, \quad \text{where } \frac{\Lambda}{\tilde{\Lambda}} = l$$

$$\Delta d = \begin{cases} (\tilde{\Lambda}/\Lambda)^{-1} - 1 \to \ln l \\ -\ln \tilde{\Lambda}/\Lambda \to \ln l \\ \frac{1}{d-2}\left[1 - \left(\frac{\tilde{\Lambda}}{\Lambda}\right)^{d-2}\right] \to \ln l \end{cases} \qquad \left(\Lambda/\tilde{\Lambda}\right)^{d-2} \simeq 1 + (d-2)\ln l$$

$$\Rightarrow \frac{1}{u} = \left[ \frac{1}{u_0} - (n-2) K_d \ln l \right] \left(1 + (d-2)\ln l\right] = \frac{1}{u_0} + \frac{(d-2)t}{u_0} - (n-2) K_d \ln l$$

$$\Rightarrow -\frac{du}{u_0^2} = \frac{1}{u_0}(d-2) K_d \ln l - (n-2) C \cdot \ln l$$

$$\boxed{\Rightarrow \frac{du}{d\ln l} = (2-d)u + K_d (n-2) u^2}$$

2-d is the naive dimension of $u$. Usullay in the $\varphi^4$-theory, interactions are $u\varphi^4$-term, thus the naive dimension of $u$ is 4-d. Now due to the constrain $|\vec{n}^2| = 1$, $\frac{1}{2g}(\partial_\mu n)^2$ is already interacting, but its dimension is 2-d. If no constraint, g cannot be interpreated as interaction strength!

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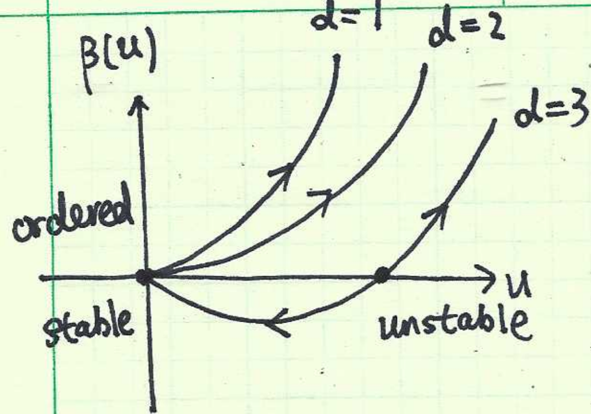
• disordered






① at $d=1,2$ and $n \geq 3$ there are only strong coupling fixed point

② at $d \geq 3$, $n \geq 3$, there is a unstable fixed point.

and there's a order-disorder transition.

we do have a long-range ordering phase at low T.

③ if $n=2$, if $d \geq 3$, there's also a low T long-range order phase. High order calcalations will give rise to a up-turn of the RG curve. At large $u$ (or large T, we will ultimatly go to disordered phase).

④ if $n=2$ and $d=2$. $\longrightarrow$ KT transition (see later lectures).

The $\beta(u)$ also provided a way to calculate the correlation length.

$$\xi[u(\Lambda), \Lambda] = \xi[u(\tilde{\Lambda}), \tilde{\Lambda}]$$

where we do $\Lambda \to \tilde{\Lambda}$, $\xi$ doesn't change, but the ratio $\xi / \Lambda^{-1}$ changes. And the change of $\Lambda$, is compensated by varing $u(\tilde{\Lambda})$.

$$\Lambda \frac{\partial}{\partial \Lambda} \xi[u(\Lambda), \Lambda] = 0 \Rightarrow \left.\frac{\partial \xi}{\partial \ln \Lambda}\right|_{\text{fix } u} + \Lambda \frac{\partial \xi}{\partial u} \cdot \frac{\partial u}{\partial \Lambda} = 0$$

dimensional analysis shows ← we do not have other parameters such as $r_0$

$$\xi[u(\Lambda), \Lambda] \sim \Lambda^{-1} \phi(u) \quad \text{if fixed } u, \ \xi \sim \Lambda^{-1}$$

thus $\boxed{\left.\frac{\partial \xi}{\partial \ln \Lambda}\right|_{\text{fix } u} = -\xi}$, and $\Lambda \frac{\partial u}{\partial \Lambda} = -\frac{\partial u}{\partial \ln l} = -\beta(u)$

$$\Rightarrow \xi(u, \Lambda) + \beta(u) \left.\frac{\partial \xi}{\partial u}(u, \Lambda)\right|_{\text{fixed } \Lambda} = 0$$

$$\Rightarrow -\frac{du}{\beta(u)} = \frac{d\xi}{\xi}$$

$$\Rightarrow \int_{\xi_0}^{\xi} \frac{d\xi}{\xi} = -\int_{u_0}^{u} \frac{du}{\beta(u)} \Rightarrow \ln\left(\frac{\xi}{\xi_0}\right) = -\int_{u_0}^{u} \frac{du}{\beta(u)}$$

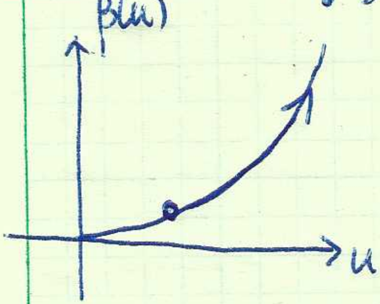
$$\text{or } \boxed{\xi = \xi_0 \exp\left[-\int_{u_0}^{u} \frac{du}{\beta(u)}\right]} \leftarrow \text{fix } \Lambda.$$

for $d=2$: $\beta(u) = \frac{(n-2)}{2\pi} u^2$

let us take $u_0 = \infty$, and at this case $\xi_0 \simeq a_0 = 1/\Lambda$

$$\xi(u) = \xi_0 \exp\left[-\int_{+\infty}^{u} \frac{2\pi\, du}{(n-2)u^2}\right] = \xi_0\, e^{\frac{2\pi}{n-2} \int_{u_0}^{+\infty} \frac{du}{u^2}}$$

$$\boxed{\xi(u) = \xi_0\, e^{\frac{2\pi}{n-2} \frac{1}{u}}}$$




• strong coupling fixed

$d=2$ non-linear $O(3)$ $\sigma$ model

– Quantum spin chain with integral spin

Haldane: $\boxed{\mathcal{L}_E = \frac{1}{2g}\left[ v_s (\partial_0 \vec{m})^2 + \frac{1}{v_s}(\partial_x \vec{m})^2 \right] + \frac{i\theta}{8\pi} \epsilon_{ij}\, \vec{m} \cdot (\partial_i \vec{m} \times \partial_j \vec{m})}$

where $g = 2/S$ , $v_S = 2a_0 JS$

The second term is called $\theta$-term.

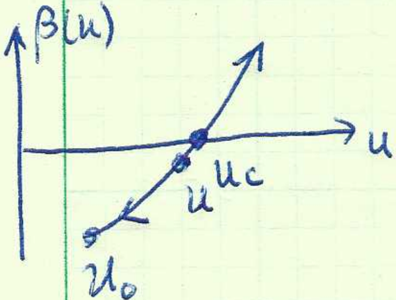
① half integer. $\theta$-term: $(-1)$ for odd winding number

Quantum effect! → criticality (Haldane conjecture).

② integer spin — usual non-linear $\sigma$-model

for $d=3$. $\beta(u) = -u + \frac{u^2}{2\pi^2}$ , and $u_c = 2\pi^2$

$\simeq (u - u_c)$ with $u_c = 2\pi^2$

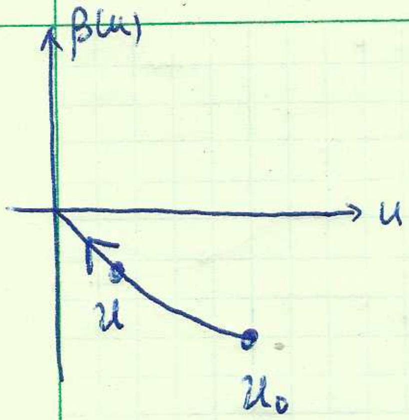



for $u \sim u_c$, we have

$$\xi(u) = \xi(u_0) \exp\left[ -\int_{u_0}^{u} \frac{du}{u-u_c} \right] \text{ as } u \to u_c$$

$$= \xi(u_0)\, e^{-\ln\left|\frac{u-u_c}{u_0-u_c}\right|}$$

$$\xi(u) = \xi(u_0) \left| \frac{u-u_c}{u_0-u_c} \right|^{-1} \text{ as } u \to u_c$$



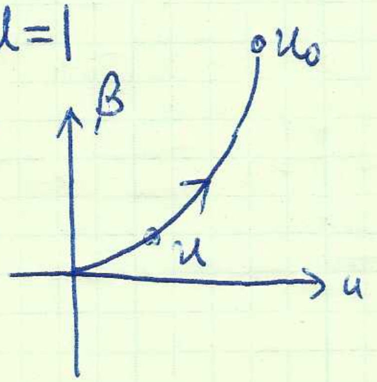

for fixed point arond $u=0$

$$\beta(u) = -u$$

$$\xi(u) = \xi(u_0)\, e^{-\int_{u_0}^{u} \frac{du}{-u}} = \xi(u_0)\, e^{\ln u/u_0}$$

$$= \xi(u_0)\frac{u}{u_0} \longrightarrow 0.$$

For $d=1$




$$\beta(u) = u$$

$$\xi(u) = \xi(u_0 \to \infty)\, e^{-\int_{u_0}^{u} \frac{du}{u}}$$

$$= \xi(u_0 \to \infty)\left(\frac{u_0}{u}\right)$$

$$\sim \frac{a_0}{u} \qquad \text{set } u_0 \sim 1$$