$4-\epsilon$ expansion: Calculation of critical exponents

Now let us calculate the fixed points, A obvious fixed point is the Gaussian one with $\left(\frac{r^*}{\Lambda^2}, \frac{u}{\Lambda^\epsilon}\right) = (0,0)$. We can linearize around $\left(\frac{r^*}{\Lambda^2}, \frac{u^*}{\Lambda^\epsilon}\right)$ as

$$A = 3K_4 = \frac{3}{8\pi^2}$$

$$\begin{matrix} \frac{d\, r/\Lambda^2}{d\ln l} = \\ \frac{d\, u\Lambda^{-\epsilon}}{d\ln l} = \end{matrix} \begin{bmatrix} 2 & A \\ 0 & \epsilon \end{bmatrix} \begin{bmatrix} r/\Lambda^2 \\ u/\Lambda^\epsilon \end{bmatrix}$$

The eigenvales of this matrix

$$\nu = 1/y_t = 1/2$$

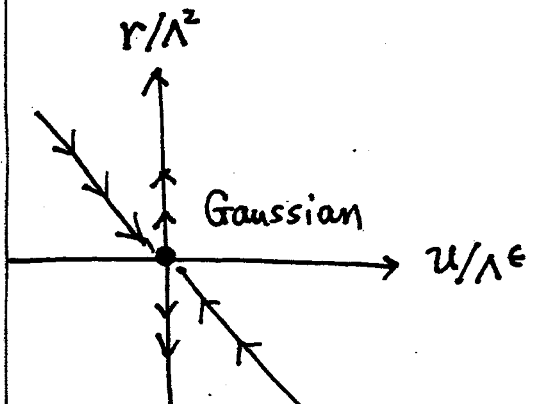
$$\Lambda_t = l^{y_t} \text{ and } y_t = 2, \text{ and the eigenvecter } \vec{V}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\Lambda_2 = l^\epsilon \text{ and } y_2 = \epsilon : \text{ eigenvevector } \vec{V}_2 = \left(-\frac{A}{2}, 1\right)$$

These two eigenvecters are not perpendicular to each other.

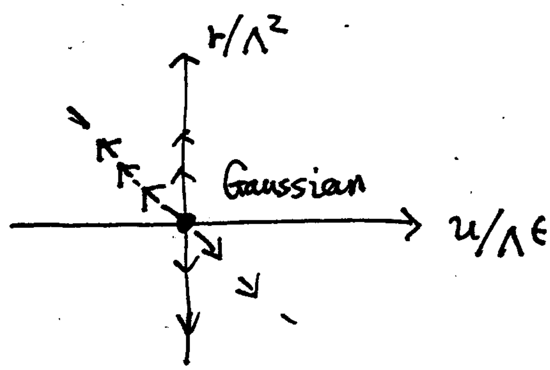
This is basically what we calculated before in Gaussian approximation, Combine the wavefunction renormalization

$$\Lambda_h = Z = l^{1+d/2} \rightarrow y_h = 1 + d/2 \Rightarrow \eta = 0$$



a) $d>4$, or $\epsilon = 4-d < 0$

Stable with respect to u



b) $d<4$ or $\epsilon > 0$

Gaussian fixed point unstable with respect to u.

Now there's a new fixed point; we can solve

$$\left.\begin{aligned}\frac{u^*/\Lambda^\epsilon}{(1+r^*/\Lambda^2)^2} &= \frac{1}{B}\epsilon \\ \frac{r^*}{\Lambda^2} &= -\frac{A}{2}\frac{u^*/\Lambda^\epsilon}{1+r^*/\Lambda^2}\end{aligned}\right\} \Rightarrow \frac{r^*}{\Lambda^2} = -\frac{\epsilon}{2}\frac{A}{B}\frac{1}{1+\frac{r^*}{\Lambda^2}}$$

$$\frac{A}{B} = \frac{3}{9} = \frac{1}{3}$$

$$\Rightarrow \frac{r^*}{\Lambda^2} = -\frac{\epsilon}{6}\frac{1}{1+(r^*/\Lambda^2)}$$

Since we have approximated to linear order of ϵ, it does help if we solve this equation rigorously, but we just take $\frac{r^*}{\Lambda^2} = -\frac{\epsilon}{6}$

Wilson -Fisher

$$\boxed{\begin{aligned} u^*/\Lambda^\epsilon &= \frac{1}{B}\epsilon = \frac{8\pi^2}{9}\epsilon \\ r^*/\Lambda^2 &= -\epsilon/6 \end{aligned}}$$

$\leftarrow$ to $O(\epsilon)$.

W-F fixed is along the direction of $\vec{V}_2$.

Linearized the RG equation at Wilson-Fisher fixed point

the non-linear version of RG

$$\frac{d\, r/\Lambda^2}{d\ln l} = 2(r/\Lambda^2) + \frac{A}{1+r/\Lambda^2}u/\Lambda^\epsilon \qquad ①$$

$$\frac{d\, u/\Lambda^\epsilon}{d\ln l} = u/\Lambda^\epsilon\left[\epsilon - \frac{B}{(1+r/\Lambda^2)^2}u/\Lambda^\epsilon\right] \qquad ②$$

① : ~~To~~ Take derivative on $(r/\Lambda^2) \Rightarrow 2 - \frac{A}{(1+r/\Lambda^2)^2}u/\Lambda^\epsilon\Big|_*$

$$= 2 - \frac{A}{(1+r^*/\Lambda^2)^2}u^*/\Lambda^\epsilon \simeq 2 - \frac{A}{B}\epsilon = 2(1-\tfrac{1}{6}\epsilon)$$

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take derivative on u/Λ^ϵ $\Rightarrow$ $\left.\frac{A}{1+r/\Lambda^2}\right|_* \simeq \frac{3}{8\pi^2}\frac{1}{1+\epsilon/6}$

$$\simeq \frac{3}{8\pi^2}\left(1-\epsilon/6\right)$$

$$\Rightarrow \frac{d\,\delta r/\Lambda^2}{d\ln l} = 2\left(1-\frac{\epsilon}{6}\right)\delta r/\Lambda^2 + \frac{3}{8\pi^2}\left(1-\epsilon/6\right)\delta u/\Lambda^\epsilon$$

Now let's look at ②

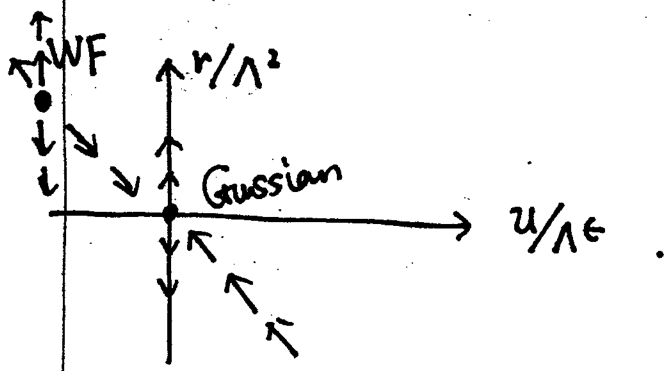
take derivative with respect to u/Λ^ϵ $\Rightarrow$ $\left.\epsilon - \frac{2B}{(1+r/\Lambda^2)^2}\,u/\Lambda^\epsilon\right|_*$

$$= \epsilon - \frac{u^*/\Lambda^\epsilon}{(1+r^*/\Lambda^2)^2}\,2B$$

$$= \epsilon - 2\epsilon = -\epsilon$$

$$\Rightarrow \frac{d\,\delta u/\Lambda^\epsilon}{d\ln l} = -\epsilon\,\delta u/\Lambda^\epsilon$$

$$\Rightarrow \begin{bmatrix} \frac{d\,\delta r/\Lambda^2}{d\ln l} \\ \frac{d\,\delta u/\Lambda^\epsilon}{d\ln l} \end{bmatrix} = \begin{bmatrix} 2\left(1-\frac{\epsilon}{6}\right), & \frac{3}{8\pi^2}\left(1-\frac{\epsilon}{6}\right) \\ 0 & -\epsilon \end{bmatrix} \begin{bmatrix} \delta r/\Lambda^2 \\ \delta u/\Lambda^\epsilon \end{bmatrix}$$



$\Lambda_t = l^{y_t}$, $y_t = 2(1-\epsilon/6)$

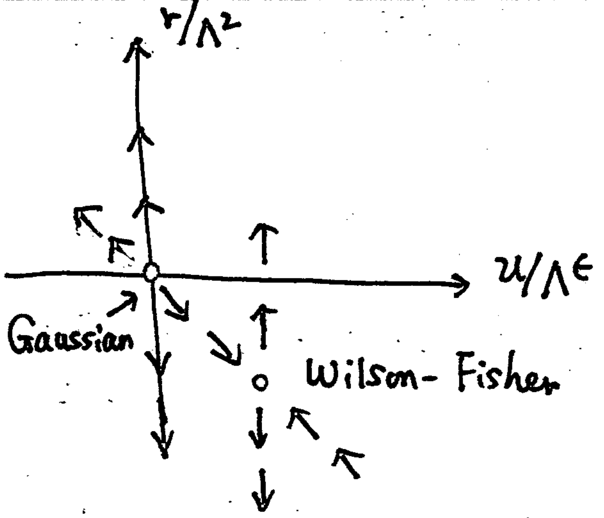
eigenvector $\vec{v}_1 = (1, 0)$

$\Lambda_2 = l^{-\epsilon}$, with

$\vec{V}_2 = \left[-\frac{3}{8\pi^2}, 2\right]$

the same as Gaussian

a) For $d>4$: the W-F fixed point lies in the region that $u<0$, and thus unphysical.



① $d<4$

W-F fixed point

finite interaction fixed point

$$\begin{cases} \nu = 1/y_t = \dfrac{1}{2(1-\epsilon/6)} = \frac{1}{2}(1+\epsilon/6) \\ \eta = 0 \quad (\text{no wave function renormalization}) \end{cases}$$

$$\alpha = 2 - \nu d$$

$$\beta = \frac{\nu}{2}(d-2+\eta)$$

$$\gamma = \nu(2-\eta)$$

$$\delta = \frac{d+2-\eta}{d-2+\eta}$$

$$\Rightarrow \begin{cases} \alpha = \epsilon/6 \\ \beta = 1/2 - \epsilon/6 \\ \delta = 3+\epsilon \\ \gamma = 1+\epsilon/6 \end{cases} \xrightarrow{\epsilon=1} \begin{matrix} 0.167 \\ 0.33 \\ 4 \\ 1.167 \end{matrix}$$

c.f 3d Ising

$$\boxed{\begin{matrix} \alpha = 0.110 \\ \beta = 0.325 \\ \gamma = 1.240 \\ \delta = 4.82 \end{matrix}}$$

not bad!