Supplemental Material — Correlation functions of Ising model

1. Quantum Ising model at critical point

\[ H = -K \sum_i \left[ g \sigma_x(i) + \sigma_z(i) \sigma_z(i+1) \right] \]

with the correspondence to the 2D classic Ising model.

\[
\begin{align*}
K = \beta J \\
\sinh(2\beta J g) \sinh(2\beta J) = 1
\end{align*}
\]

\[
\begin{align*}
\sinh(z\alpha \gamma) \sinh(z\beta J) = 1
\end{align*}
\]

\[
\begin{align*}
\text{from } K g = \hbar \alpha \\
\text{at the critical } T_c \text{ of 2D Ising model, } \sinh(2\beta_c J) = 1
\end{align*}
\]

\Rightarrow \quad g_c = 1 \text{ at quantum critical point of 1D quantum Ising model.}

2. Define a Jordan-Wigner transformation

\[
\begin{align*}
\xi_1(n) &= \frac{1}{\sqrt{2}} \left( \prod_{i<n} \sigma_x(i) \right) \sigma_y(n) \\
\xi_2(n) &= \frac{1}{\sqrt{2}} \prod_{i<n} \sigma_x(i) \sigma_z(n)
\end{align*}
\]

\[
\{ \xi_i(n), \xi_j(m) \} = \delta_{ij} \delta_{mn}
\]
(1) If \( n \neq m \)

Proof: Suppose \( n < m \), then \( a \cdot b = y \) in \( \mathbb{Z} \)

\[
\Omega \left( \frac{\delta_i(n)}{\delta_j(m)} \right) = \frac{1}{x} \left[ \prod_{l \leq n} \sigma_x(l) \right] \sigma_a(n) \left[ \prod_{l' \leq m} \sigma_x(l') \right] \sigma_b(m)
\]

\[
= \frac{1}{2} \left[ \prod_{l \leq n} \sigma_x(l) \right] \left[ \prod_{l' \leq m} \sigma_x(l') \right] (-) \sigma_a(n) \sigma_b(m)
\]

\[
= -\frac{1}{2} \left[ \prod_{l \leq m} \sigma_x(l) \right] \left[ \prod_{l \leq n} \sigma_x(l) \right] \sigma_b(m) \sigma_a(n)
\]

\[
= -\frac{1}{2} \left[ \prod_{l \leq m} \sigma_x(l) \right] \sigma_b(m) \left[ \prod_{l \leq n} \sigma_x(l) \right] \sigma_a(n) = -\frac{\delta_j(m)}{\delta_i(n)} \delta_{ij}
\]

(2) If \( n = m \)

\[
\delta_i(n) \delta_j(n) = \frac{1}{2} \left[ \prod_{l \leq n} \sigma_x(l) \right] \sigma_a(n) \left[ \prod_{l' \leq n} \sigma_x(l') \right] \sigma_b(n)
\]

\[
= \frac{1}{2} \left[ \prod_{l \leq n} \sigma_x(l) \right] ^2 \sigma_a(n) \sigma_b(n) = \frac{1}{2} \sigma_a(n) \sigma_b(n)
\]

\[
\Rightarrow \delta_i(n) \delta_j(n) + \delta_j(n) \delta_i(n) = \delta_{ij}
\]

Combine together \( \Rightarrow \{ \delta_i(n), \delta_j(n) \} = \delta_{ij} \delta_{mn} \)

and \( \Omega_i^2(n) = \Omega_j^2(n) = \frac{1}{2} \).

Majorana fermion operator

(\#) Some relations

\[
\sigma_x(2\ell) = -i \sigma_y (\ell) \sigma_z (2\ell) = -2i \delta_i (n) \delta_j (n)
\]
\[
\xi(n)\xi(n+1) = \frac{1}{2} \left( \prod_{i<n} \sigma_x(i) \sigma_y(n) \right) \left( \prod_{i<n+1} \sigma_x(i') \sigma_z(n+1) \right) \\
= \frac{1}{2} \left( \prod_{i<n} \sigma_x(i)^2 \sigma_y(n) \sigma_x(n) \sigma_z(n+1) \right) = -\frac{\alpha}{2} \sigma_z(n) \xi(n+1)
\]

Set \( K = \frac{1}{2} \), \( \Delta Z = 2\beta J \)

\[
H = -\frac{1}{2} \sum_i \left[ g \sigma_x(i) + \sigma_z(i) \sigma_z(i+1) \right] \\
= \sum_i \left[ i g \xi_1(i) \xi_2(i) - i \xi_1(i) \xi_2(i+1) \right]
\]

\[
H = \frac{1}{2} \left( 1 - g \right) \sum_i \left[ -i \xi_1(i) \xi_2(i) + i \xi_2(i) \xi_1(i) \right] \\
+ \frac{1}{2} \sum_i \left[ \xi_1(i) \left( \xi_2(i+1) - \xi_2(i) \right) - i \xi_2(i) \left( \xi_1(i) - \xi_1(i-1) \right) \right]
\]

\[
\frac{1}{2} \int dx \left[ \frac{1-g}{a} \left( \xi_1(x), \xi_2(x) \right) \left( 1, -i \right) \left( \xi_1 \right) \right] \\
+ \frac{1}{2} \int dx \left( \xi_1, \xi_2 \right) \left( -i \partial_x \right) \left( \xi_1 \right)
\]

\[
H = \frac{1}{2} \int dx \xi^T \left( \alpha P + \beta m \right) \xi, \quad \text{with} \quad m = \frac{1-g}{a} \\
\left\{ \begin{array}{l}
\beta = \sigma_z, \quad \alpha = \sigma_1
\end{array} \right.
\]
Consider the correlation of $\sigma$ in the 2D Ising model along the same row

$$\langle \sigma_{ij}, \sigma_{i'j'} \rangle = \frac{\sum_{\{\sigma\}} \sigma_{ij} \sigma_{i'j'} e^{\beta J \sum \sigma_{mn} \sigma_{mn'}}}{\sum_{\{\sigma\}} e^{\beta J \sum \sigma_{mn} \sigma_{mn'}}}$$

$$= \frac{\sum \text{Tr} \left[ \sigma_z(i) \sigma_z(i') T^N \right]}{\sum \text{Tr} \left[ T^N \right]}$$

where $T = e^{\hat{h} \beta} = e^{\frac{\beta}{2} \sum_{i} \sigma_x(i) + \beta J \sum \sigma_z(i) \sigma_z(i+1)}$

with $\sinh (k \beta) \sinh 2 \beta J = 1$.

$$G(i,j) = \frac{\sum \langle a | T^N | b \rangle \langle b | \sigma_z(i) \sigma_z(j) | a \rangle}{\sum \langle a | T^N | a \rangle}$$

as $N \to \infty$, we only need to consider the ground state $|\psi\rangle$ of $T$, i.e. the largest eigenvalue.
\[ G(i,j) \xrightarrow{N \to \infty} \langle 0| \sigma_z(i) \sigma_z(j) | 0 \rangle = \langle 0| \sigma_z(i) \sigma_z(i+1) \cdots \sigma_z(j-1) \sigma_z(j) | 0 \rangle \]

\[ = \langle 0| \prod_{l=i}^{j-1} \sigma_z(l) \sigma_z(l+1) | 0 \rangle = \langle 0| (\mathbb{S} \mathbb{S}) \left( \prod_{l=i+1}^{j} \mathbb{S}_1(l) \mathbb{S}_2(l) \right) \mathbb{S}_2(j) | 0 \rangle \]

It can be expressed in terms of Pfaffian, but it's too complicated.

Consider two copies of Ising model \( \sigma' \) and \( \sigma'' \), represented by \( \mathbb{S}_1 \) and \( \mathbb{S}_2 \), respectively.

We calculate \( \langle \sigma(i) \sigma_c(i) \sigma(j) \sigma(j) \rangle = \langle \sigma(i) \sigma(j) \rangle \langle \sigma_c(i) \sigma(j) \rangle = G^2(i,j) \)

\[ G^2(i,j) = \langle o_0 \otimes o_0 | (2 \mathbb{S})^{\frac{i}{2}} \left( \prod_{l=i+1}^{j-1} 2 \mathbb{S}_1(l) \mathbb{S}_2(l) \right) \mathbb{S}_2(j) \otimes (\mathbb{S} \to \mathbb{S}) | o_0 \otimes o_0 \rangle \]

\[ = \langle o_0 \otimes o_0 | 2 \mathbb{S}_1(i) \mathbb{S}_2(i) \left( \prod_{l=i+1}^{j-1} 2 \mathbb{S}_1(l) \mathbb{S}_2(l) \right) 2 \mathbb{S}_2(j) \mathbb{S}_2(j) | o_0 \otimes o_0 \rangle \]

Since \( (2 \mathbb{S})^2 = -1 \) \( \Rightarrow 2 \mathbb{S} = e^{\frac{\pi}{2} (2 \mathbb{S})} = 2 \mathbb{S} \)

\[ G^2(i,j) = \langle o_0 \otimes o_0 | 2 \mathbb{S}_1(i) \mathbb{S}_2(i) \exp \left[ i \pi \sum_{l=i+1}^{j-1} \mathbb{S}_1(l) \mathbb{S}_1(l) \mathbb{S}_2(l) + \mathbb{S}_2(l) \mathbb{S}_2(l) \right] \]

\[ - 2 \mathbb{S}_2(i) \mathbb{S}_2(i) | o_0 \otimes o_0 \rangle \]
Recall $H_3 = \int dx \left\{ \frac{i}{2} \left[ s_1 \partial x s_2 + s_2 \partial x s_1 \right] - \text{im} (s_1 s_2) \right\}$

define chiral basis $s_{R,L} = \frac{s_1 \pm s_2}{\sqrt{2}}$

$\Rightarrow H_3 = \int dx \left\{ \frac{i}{2} (s_R \partial x s_R - s_L \partial x s_L) + \text{im} s_R s_L \right\}$

Double the Hamiltonian, define $\mathcal{H}_{R,L} = \frac{n_1 \pm n_2}{\sqrt{2}}$

$H_2 = \int dx \left\{ \frac{i}{2} (n_R \partial x n_R - n_L \partial x n_L) + \text{im} n_R n_L \right\}$

$H = H_3 + H_2 = \int dx \{ -\frac{i}{2} [s_R \partial x s_R + n_R \partial x n_R - (R \rightarrow L)] + \text{im} (s_R s_L) \}$

Now define $\psi_{R,L} = \frac{s_{R,L} + i n_{R,L}}{\sqrt{2}}$, then

$\psi_R^+ \partial x \psi_R = \frac{1}{2} \left[ s_R - in_R \right] \partial x \left[ s_R + i n_R \right] = \frac{1}{2} \left[ s_R \partial x s_R + n_R \partial x n_R \right]$ ... 

$\psi_R^+ \psi_L = \frac{1}{2} \left[ s_R - in_R \right] \left[ s_L + i n_L \right] = \frac{1}{2} \left[ s_R s_L + n_R n_L + i (s_R n_L - n_R s_L) \right]$

$\psi_L^+ \psi_R = \frac{1}{2} \left[ s_L - in_L \right] \left[ s_R + i n_R \right] = \frac{1}{2} \left[ s_L s_R + n_L n_R + i (s_L n_R - n_L s_R) \right]$ 

$H = \int dx \left\{ \psi_R^+ (-i \partial x) \psi_R + \psi_L^+ (i \partial x) \psi_L + \text{im} (\psi_R^+ \psi_L - \psi_L^+ \psi_R) \right\}$
Then \( S_{1,2} = \frac{1}{\sqrt{2}} (s_R \pm s_L) \) \( \Rightarrow \ s_1 n_1 = \frac{1}{2} (s_R n_R + s_L n_L + s_R n_L + s_L n_R) \)
\[ n_{1,2} = \frac{1}{\sqrt{2}} (n_R \pm n_L) \]
\( s_2 n_2 = \frac{1}{2} (s_R n_R + s_L n_L - s_R n_L - s_L n_R) \)
\[ \Rightarrow \ s_1 n_1 + s_2 n_2 = s_R n_R + s_L n_L = \frac{1}{\sqrt{2}i} (\psi_R^+ + \psi_L^+)(\psi_R^- - \psi_L^-) = \frac{1}{2i} \left[ \psi_R^+ \psi_R^- + \psi_L^+ \psi_L^- \right] \]
\[ + \frac{1}{\sqrt{2}i} (\psi_R^+ + \psi_L^+)(\psi_R^- - \psi_L^-) \]
\[ = \frac{1}{2} \left[ \psi_R^+ \psi_R^- - \frac{1}{2} \right] + \frac{1}{2} \left[ \psi_L^+ \psi_L^- - \frac{1}{2} \right] \]
\[ \Rightarrow \ : \psi_R^+ \psi_R^- + \psi_L^+ \psi_L^- : = i s_1 n_1 + i s_2 n_2 \]
\[ s_R n_L + s_L n_R = \frac{1}{\sqrt{2}i} (\psi_R^+ + \psi_L^+)(\psi_R^- - \psi_L^-) = \frac{1}{2i} \left[ \psi_R^+ \psi_R^- + \psi_L^+ \psi_L^- \right] \]
\[ = \frac{1}{2i} \left[ -\psi_R^+ \psi_L^- + \psi_L^+ \psi_R^- + \psi_R^+ \psi_L^- - \psi_L^+ \psi_R^- \right] = \frac{1}{2i} \left[ \psi_R^+ \psi_L^- + \psi_L^+ \psi_R^- \right] \]
\[ \Rightarrow i s_1 n_1 = \frac{1}{2} : \psi_R^+ \psi_R^- + \psi_L^+ \psi_L^- : + \frac{1}{2} \left[ \psi_R^+ \psi_L^- + \psi_L^+ \psi_R^- \right] \]
\[ i s_2 n_2 = \frac{1}{2} : \psi_R^+ \psi_R^- + \psi_L^+ \psi_L^- : - \frac{1}{2} \left[ \psi_R^+ \psi_L^- + \psi_L^+ \psi_R^- \right] \]

\* Bosonization of the model

\[ \psi_R(x) = \frac{1}{\sqrt{2\pi a}} e^{i \sqrt{4\pi} \phi_R(x)} \]
\[ \psi_L(x) = \frac{1}{\sqrt{2\pi a}} e^{-i \sqrt{4\pi} \phi_L(x)} \]
\[ [\phi_R(x), \phi_R(x')] = \frac{i}{4} \text{sgn}(x-x') \]
\[ [\phi_L(x), \phi_L(x')] = -\frac{i}{4} \text{sgn}(x-x') \]
\[ [\phi_R(x), \phi_L(x')] = \frac{i}{4} \]
\[ \phi(x) = \phi_R(x) + \phi_L(x), \quad \Theta(x) = \phi_R(x) - \phi_L(x) \]

\[ [\phi(x), \phi(x')] = [\Theta(x), \Theta(x')] = 0 \]

\[ [\phi(x), \Theta(x')] = -i \Theta(x' - x) = \begin{cases} 
0 & x' < x \\
-i & x' > x 
\end{cases} \]

*bosonic variable*

\[ p_R = :\psi_R^+(x+\epsilon)\psi_R(x): = \lim_{\epsilon \to 0} \lim_{a \to 0} \left( \psi_R^+(x+\epsilon)\psi_R(x) - \langle \psi_R^+(x+\epsilon)\psi_R(x) \rangle \right) \]

\[ \psi_R^+(x+\epsilon)\psi_R(x) = \frac{1}{2\pi a} e^{-i\sqrt{4\pi} \phi_R(x+\epsilon)} e^{i\sqrt{4\pi} \phi_R(x)} \]

\[ e^A e^B = :e^{A+B}: e^{\langle AB + \frac{A^2 + B^2}{2} \rangle} \]

\[ = :e^{-i\sqrt{4\pi} \epsilon \partial_x \phi_R}: e^{4\pi \langle \phi_R(x+\epsilon)\phi_R(x) - \phi_R^2(0) \rangle} \]

\[ \langle \phi_R(x)\phi_R(x') \rangle = \frac{-1}{4\pi} \ln \frac{2\pi}{L} \left[ a - i(x-x') \right] \]

\[ \Rightarrow e^{4\pi \langle \phi_R(x+\epsilon)\phi_R(x) - \phi_R^2(0) \rangle} = \frac{a}{a-i\epsilon} \]

\[ \Rightarrow p_R = \lim_{\epsilon \to 0} \lim_{a \to 0} \frac{1}{2\pi a} \left[ i e^{-i\sqrt{4\pi} \epsilon \partial_x \phi_R} - 1 \right] \frac{a}{a-i\epsilon} \]

\[ = \lim_{\epsilon \to 0} \lim_{a \to 0} \frac{1}{2\pi a} \left( -i\sqrt{4\pi} \epsilon \partial_x \phi_R \right) \frac{a}{a-i\epsilon} = \sqrt{\frac{i}{4\pi}} \partial_x \phi_R(x) \]

Similarly

\[ p_L = \sqrt{\frac{i}{4\pi}} \partial_x \phi_L(x) \]

\[ p(x) = :\psi_R^+(x)\psi_R(x) + \psi_L^+\psi_L(x): = \sqrt{\frac{i}{4\pi}} \partial_x \phi \]
\[ \psi_R \psi_L = \frac{1}{2\pi a} e^{i\sqrt{\pi} \phi_R} e^{-i\sqrt{\pi} \phi_L} = \frac{1}{2\pi a} e^{-i\sqrt{\pi} \phi} e^{-\frac{4\pi}{\sqrt{\pi}} [\phi_R, \phi_L]} = \frac{1}{2\pi a} e^{-i\sqrt{\pi} \phi} e^{-\frac{i}{4} \frac{1}{\sqrt{\pi}}} = \frac{i}{2\pi a} e^{i\sqrt{\pi} \phi} \]

\[ \psi_L^\dagger \psi_R = \frac{i}{2\pi a} e^{-i\sqrt{\pi} \phi} \]

\textit{Apply the above result}

\[ \exp \left[ i\pi \sum_{i,j} (s_1(i) \eta_1(i) + s_2(i) \eta_2(i)) \right] = e^{i\pi \int \delta x \partial_x \phi} \]

\[ = e^{i\sqrt{\pi} \phi(j-a)} e^{-i\sqrt{\pi} \phi(i+a)} \]

\[ i s_1(i) \eta_1(i) = \frac{a}{2} \left[ \partial_x \phi(i) + \frac{i}{2} \frac{-i}{2\pi a} e^{i\sqrt{\pi} \phi(i)} + \frac{i}{2\pi a} e^{-i\sqrt{\pi} \phi(i)} \right] \]

\[ i s_2(j) \eta_2(j) = \frac{a}{2} \left[ \partial_x \phi(j) - \frac{i}{2} \frac{-i}{2\pi a} e^{-i\sqrt{\pi} \phi(j)} + \frac{i}{2\pi a} e^{i\sqrt{\pi} \phi(j)} \right] \]

\textit{The leading contribution is}

\[ G^2(i,j) = \frac{1}{(4\pi)^2} \frac{i (-i)}{C_{\sigma \sigma}} e^{i\sqrt{\pi} \phi(i)} e^{-i\sqrt{\pi} \phi(j)} e^{i\sqrt{\pi} \phi(i)} e^{-i\sqrt{\pi} \phi(j)} \]

\[ \sim \frac{1}{(4\pi)^2} \frac{1}{C_{\sigma \sigma}} e^{i\sqrt{\pi} \phi(i)} e^{-i\sqrt{\pi} \phi(j)} \left< 0 \right| \phi \left| 0 \right> \]
Vertex operator
\[
\langle 0 | e^{i \beta \phi(x+t)} e^{-i \beta \phi(0)} | 0 \rangle = \left( \frac{a}{a - i(x-vt)} \right)^{\frac{\beta^2}{4\pi}} \left( \frac{a}{a + i(x+vt)} \right)^{\frac{\beta^2}{4\pi}} 
\]

\[ t = 0 \rightarrow \left[ \frac{a^2}{\alpha^2 + \chi^2} \right]^{\frac{\beta^2}{4\pi}} \]

\[ \Rightarrow G^2(i,j) \sim \left( \frac{a^2}{\chi^2} \right)^{\frac{1}{4}} \sim \frac{1}{\chi^{1/2}} \iff x = |i-j| \]

\[ \Rightarrow G(i,j) \sim \frac{1}{\chi^{d-2+\eta}} = \frac{1}{\chi^{1/4}} \]

\[ \Rightarrow \text{anomalous dimension of 2D Ising model: } \gamma = 1/4. \]

How about away from the critical point?

\[
\sinh (2(\beta + \Omega)) \sinh (2(\beta + \Omega)J) = 1 \\
\left[ \sinh (2\beta J) + \cosh (2\beta J) [2\beta J + 2 \beta \Omega J] \right] \left[ \sinh 2\beta \Omega + \cosh 2\beta \Omega \right] \\
= 1 \\
\Rightarrow 2 \sinh (2\beta J) \cosh 2\beta J \cdot 2\beta J + \cosh 2\beta J \cdot (2\beta \Omega J) = 0 \\
\Rightarrow \Delta g = -2 \sinh (2\beta J) \frac{\delta \beta}{\beta c} = +2 \sinh (2\beta J) \frac{\delta T}{T_c} \]
The quantum 1D model \( m \sim 4g \sim 4T \)

if this mass term is responsible for the exponential decay of the magnetic correlation \( g \sim \frac{1}{m} \sim \frac{1}{4T} \).

\( \Rightarrow \) \( \nu = 1 \), rather than the mean field value \( \nu = \frac{1}{2} \).

\( \Rightarrow \) 2D Ising model \( z = \frac{1}{4}, \nu = 1 \)

From scaling relation \( \Rightarrow \) \( \alpha = 0 \)

\[ \begin{cases} 
\beta = \frac{1}{8} \\
\gamma = \frac{1}{4} \\
\delta = 1.5 
\end{cases} \]
Kink - operator - Majorana operator

$$\sigma_x(n) = -i \sigma_y(n) \sigma_z(n) = -2i \xi_1(n) \xi_2(n)$$

$$\sigma_y(n) = \sqrt{2} \xi_1(n) \prod_{i < n} \sigma_x(i) = \sqrt{2} \xi_1(n) \prod_{i < n} (-2i) \xi_1(i) \xi_2(i)$$

$$\sigma_z(n) = \sqrt{2} \xi_2(n) \prod_{i < n} (-2i) \xi_1(i) \xi_2(i)$$

$$\mu_{n+1/2}^z = \prod_{j < n} \sigma_j^x = \prod_{j < n} (-2i \xi_1(j) \xi_2(j))$$

$$\mu_{n+1/2}^x = \sigma_n \sigma_{n+1} = 2i \xi_1(n) \xi_2(n+1)$$

$$\mu_{n+1/2}^y = -i \mu_{n+1/2}^z \mu_{n+1/2}^x = \prod_{j < n-1} (-2i \xi_1(j) \xi_2(j)) \prod_{j < n} (-2i \xi_1(j) \xi_2(j)) = \sqrt{2} \xi_2(n)$$

$$\sigma_n^2 \mu_{n-1/2}^z = \sqrt{2} \xi_2(n) \prod_{i < n} (-2i) \xi_1(i) \xi_2(i) \prod_{j < n} (-2i \xi_1(j) \xi_2(j)) = \sqrt{2} \xi_2(n)$$

$$\otimes \xi_2(n) = \frac{1}{\sqrt{2}} \sigma_n^2 \mu_{n-1/2}^z$$

Similarly $$\xi_1(n) = \frac{1}{\sqrt{2}i} \sigma_n^2 \mu_{n+1/2}^x$$

$$\sigma_n^2 \xi_2(n) = \frac{1}{\sqrt{2}} \mu_{n-1/2}^z$$,  $$\sigma_n^2 \xi_1(n) = \frac{1}{\sqrt{2}i} \mu_{n+1/2}^x$$

$$\mu_{n-1/2}^z \xi_2(n) = \frac{1}{\sqrt{2}} \sigma_n^2$$,  $$\mu_{n+1/2}^x \xi_1(n) = \frac{i}{\sqrt{2}} \sigma_n^2$$