dimensional analysis and anomalous dimensions

Consider a Landau theory $\beta \mathcal{L} = \int d^d r \ \frac{1}{2}\gamma(\nabla m)^2 + \alpha t m^2 + \frac{\beta}{4} m^4 - hm$

$$Z = \int Dm \ e^{-\beta \mathcal{L}}.$$

Rescale: $\phi = \gamma^{1/2} m, \ \cdots\cdots$, we arrive at the standard form

$$\boxed{Z = \int D\phi \ e^{-F(\phi)}, \text{ with } F(\phi) = \int d^d r \ \tfrac{1}{2}(\nabla\phi)^2 + \tfrac{1}{2} r_0 \phi^2 + \tfrac{1}{4} u_0 \phi^4.}$$

① Now we identify the dimensions of quantities of . in $F(\phi)$.

$[F] = 1 \quad \Rightarrow 2[\phi] \ \ L^{-2} = L^{-d} \quad \Rightarrow \quad [\phi] = L^{1-d/2},$

$[r_0][\phi]^2 = L^{-d} \quad \Rightarrow \quad [r_0] = L^{-d}\left(L^{d/2-1}\right)^2 = L^{-2},$

similarly $\quad \Rightarrow \quad [u_0] = L^{d-4}.$

② Define length scale $L \triangleq \xi(T) \triangleq r_0^{-1/2}$

and dimensionless variables

$$\varphi \equiv \frac{\phi}{\xi^{1-d/2}}, \quad x \equiv \frac{r}{\xi}, \quad \bar{u}_0 \equiv \frac{u_0}{\xi^{d-4}}, \quad \xi = r_0^{-1/2}$$

then

$$Z = \int D\varphi \ e^{-F(\varphi)} \quad \text{with}$$

$$F(\varphi) = \int d^d x \ \frac{1}{2}(\nabla_{\vec{x}} \varphi)^2 + \frac{1}{2}\varphi^2 + \bar{u}_0 \varphi^4.$$

The last term is the interaction term. If $\bar{u}_0 \ll 1$, let us use perturbation theory

$$\mathcal{Z} = \int D\varphi \, e^{-F_0(\varphi)} \, e^{-F_{int}} , \quad \text{where} \quad F_0(\varphi) = \int d^d x \, \frac{1}{2}(\nabla_x \varphi)^2 + \frac{1}{2}\varphi^2$$

$$F_{int} = \int d^d x \, \bar{u}_0 \varphi^4 .$$

$$= \int D\varphi \, e^{-F_0} \left(1 - F_{int} + \frac{1}{2!}(F_{int})^2 + \cdots\right)$$

ⓐ However, if $d<4$, as $T \to T_c$, or $t \to 0$, $\xi \to \infty$, and $\bar{u}_0 \to \infty$, thus perturbation theory fails

ⓑ if $d>4$, as $t \to 0$, $\bar{u}_0 \to 0$, thus the perturbation theory becomes more and more accurate. $\longrightarrow$ Upper critical dimension $d=4$!

ⓒ We have another way to derive $\underbrace{\quad}_{\text{Ginzburg}}$ criterion.

The critical region is entered at $\bar{u}_0(t) \simeq 1$.

$$\frac{u_0}{\xi^{d-4}} > 1. \qquad \frac{\xi^2(t)}{\xi^2(1)} = |t|^{-1}$$

$$\Rightarrow t^{\frac{4-d}{2}} < \frac{u_0}{\xi^{d-4}(1)} = u_0 \, \xi^{+4}(1) / \xi^{d}(1) = \frac{u_0 \gamma^2}{\alpha^2} \frac{1}{\xi^d(1)}$$

$$= \frac{\beta}{\alpha^2} \frac{1}{\xi^d(1)} = \frac{1}{\Delta C \, \xi^d(1)}$$

This is just the Ginzburg criterion.

ΔC is the mean field specific heat jump.

☆ The longer $\xi(1)$ and the larger ΔC, the more difficult to enter the critcal region

§ The origin of anormalous dimensions.

Let us look the two-point correlation function $G(r) = \langle \phi(r)\, \phi(r') \rangle$.

The dimension of $[\phi] = L^{1-d/2} \Rightarrow [G(r)] = L^{2-d}$. It's Fourier

transform $G(k) = \int d^d r\ G(r) e^{-ikr} \Rightarrow [G(k)] = L^2$.

This is consistent with

$$G(k) = \frac{1}{k^2 + r_0} .$$

Check: Suppose that we rescale the length unit by a factor of ϵ: $L' \equiv \epsilon L$,

Then

$$\boxed{\begin{array}{lll} G' L'^2 = G L^2 & \Rightarrow & G'(k', r_0') = G(k, r_0)\epsilon^{-2} \\ k' L'^{-1} = k L^{-1} & & \text{with } k' = \epsilon k \\ r_0' L'^{-2} = r_0 L^{-2} & & \qquad r_0' = \epsilon^2 k_0 \end{array}}$$

Q1: However, we defined before at T_c, $G(k, T_c) \sim k^{-2+\eta}$, which

does not agree with above scaling! How can it be?
at $\eta \neq 0$

Q2: the correlation length $\xi \sim r_0^{-1/2} \sim t^{-1/2}$ (because $[\xi] = L$, $[r_0] = L^{-2}$).

How can it be if we have $\nu \neq 1/2$.

> Any critical exponents rather than mean-field one, means a voilation of simple scaling dimension!

The reason is that we have also the microscopic length scale a, in addition to the correlation length scale ξ. Conside the following form

$$G(k, T_c, a) \sim a^{\eta} k^{-2+\eta}$$

$a \sim \Lambda^{-1}$
Λ is the momentum cut off

then $G'(k', T_c, a') = \epsilon^{2} G(k, T_c, a)$.

This can be term as the ϕ has an anomalous dimension $\eta/2$.

Similarly, we should have

$$\xi = r_0^{-1/2} f(r_0 a^2).$$

near $t \to 0$, assume $f(x) \sim x^{\theta}$ as $x \to 0$, we have

$$\xi(t \to 0) \sim t^{-1/2+\theta} a^{2\theta},$$ and thus $\nu = 1/2 - \theta$.