Gaussian model and Ginzburg criterion

Let's only keep the quadratic term of the Landau free energy at $T > T_c$.

$$Z = \int Dm \ e^{-F(m)}$$

where

$$F(m) = \int d^d x \ \left( (\nabla m)^2 + \frac{\alpha(T)}{2} m^2 \right) - m(x) h(x)$$

$$F(m) = \sum_k m(k) m(-k) \left[ \frac{\alpha}{2} k^2 + \frac{\alpha(T)}{2} \right] - m(k) h(-k)$$

$$= \sum_k \left[ m(k) - \frac{h(k)}{\partial k^2 + \alpha} \right] \left[ m(-k) - \frac{h(-k)}{\partial k^2 + \alpha} \right] \left[ \frac{\alpha}{2} k^2 + \frac{\alpha(T)}{2} \right]$$

$$- \sum_k \frac{1}{2} \frac{h(k) h(-k)}{\partial k^2 + \alpha}$$

denote $m'(k) = m(k) - \frac{h(k)}{\partial k^2 + \alpha}$. Because $m$ and $h$ are real

field, $\Rightarrow m(k) = m(-k)$. $\Rightarrow \sum_k m'(k) m(-k) \frac{\alpha}{2} k^2 + \alpha = \sum_k m'(k) m(k) \left( \frac{\alpha}{2} k^2 + \alpha \right)$

$$= \sum_k \left[ \text{Re} \ m(k) \right]^2 \left( \frac{\alpha}{2} k^2 + \alpha \right) + \sum_k \text{Im} \ m(k) \left( \frac{\alpha}{2} k^2 + \alpha \right)$$

The Gaussian integral $\sum'$ means summation over half of momentum space.

$$\int Dm \ e^{-F(m,h)} = e^{-\sum_k \frac{1}{2} \frac{h(k) h(-k)}{\partial k^2 + \alpha} \left[ \prod_k \sqrt{\frac{\pi}{\partial k^2 + \alpha}} \right] \left[ \prod_k \sqrt{\frac{\pi}{\partial k^2 + \alpha}} \right] \sum' \text{Re} \ m(k) \text{Im} \ m(k)} e^{-\text{Im} (m'(k)^2 (\partial k^2 + \alpha))}$$

$$= \text{Const} \ e^{-\sum_k \frac{1}{2} \frac{h(k) h(-k)}{\partial k^2 + \alpha} \left[ \prod_k \sqrt{\frac{\pi}{\partial k^2 + \alpha}} \right] \left[ \prod_k \sqrt{\frac{\pi}{\partial k^2 + \alpha}} \right]}$$

$$\Rightarrow \ln Z = -\frac{1}{2} \sum_k \ln \left( \frac{\alpha}{2} k^2 + \alpha \right) - \frac{1}{2} \sum_k \frac{h(k) h(-k)}{\partial k^2 + \alpha}$$
\[ \int_0^{\Lambda} \frac{d\vec{k}}{(2\pi)^d} \frac{1}{(\sqrt{k^2 + \alpha})^2} = \int_0^{\Lambda} \frac{d\vec{k}}{(2\pi)^d} \frac{k^{d-1}}{\alpha^2 \left( \frac{1}{k^2} + 1 \right)} \propto \frac{\alpha^{d-4}}{\alpha_0^4} \int_0^{\Lambda'} \frac{ds}{(s^2 + 1)^{d-1}} \]

with
\[ \Lambda' = \frac{\Lambda}{\sqrt{\alpha_0^2 + t}} \]

\[ \Rightarrow \text{at } d < 4, \text{we can extract the infrared divergence} \]
\[ \frac{C}{V} \propto \text{const.} \cdot \frac{t^{d-4}}{2} \]

\[ \Rightarrow \alpha = \frac{4-d}{2} \]
check case 1: as $t \to 0$, \[ \frac{C}{V} \propto \int_0^\infty \frac{d^2k}{(2\pi)^d} \frac{1}{k^4} \propto \Lambda^d \text{dependent on constant.} \]

check case 2: \[ \frac{C}{V} \propto \int_0^\infty \frac{k^2 dk}{(k^2 + \alpha^2)^3} \propto \int_0^\infty \frac{k^2 dk}{(k^2 + \alpha^2)^2} \]

set $y = k^2 + \alpha^2$ \[ \Rightarrow \frac{C}{V} \propto \int_0^{\Lambda^2} \frac{(y - \alpha^2)}{y^2} dy \sim \ln \frac{\Lambda^2}{\alpha^2} \sim \text{ultraviolet} \]

The above result shows that we cannot neglect the Gaussian fluctuation at $d < 4$, and we also need to be careful at $d = 4$. At $d > 4$, the ultraviolet divergence does not affect the phase transition.
The GL theory fails when the fluctuations is strong.

Define: \( \frac{1}{V} \int_\xi dr \, m(r) \). The integral over the size of correlation length \( \xi \).

The long range order \( \bar{m} = \frac{1}{V} \langle \int_\xi dr \, m(r) \rangle = \frac{\alpha_0}{B} |t| \)

Define \( \frac{1}{8\pi} |t| \int_\xi dr \left\{ \langle m(0) m(r) \rangle - \bar{m}^2 \right\} = \frac{1}{8\pi} |t| \int_\xi dr \, G(r) = E_{GL} \)

which is a characteristic quantity to judge the fluctuation effect.

The denominator: \( \bar{m}^2 = \frac{-\alpha(T)}{B} = \frac{\alpha_0}{B} |t| \)

\( \frac{2}{\xi(t)} = \frac{\xi}{\alpha_0} |t|^{-1} = \xi(1) |t|^{-1} \)

where \( \xi(1) \) is the correlation length far away free the critical region. ( \( \xi(1) = \sqrt{\frac{\xi}{\alpha_0}} \).)

\( \Rightarrow \) denominator \( \frac{\alpha_0}{B} \xi(1) |t|^{1-d/2} \)

The numerator \( \int_\xi dr \, G(r) \leq k_B T \chi_T \approx k_B T \frac{1}{4\alpha_0 |t|} \)
$$E_{GL} = \frac{k_B T_c}{4 \alpha_0 \beta t} \frac{\beta}{\alpha_0 \beta t^{1-d/2}} = \frac{k_0}{4 \Delta C} \frac{1}{t^{1-\frac{d}{2}}} = k_0 t^{-\frac{d}{2}}$$

where $\Delta C = \frac{\alpha_0^2}{\beta} T_c$ is the mean field specific heat jump at the transition. If $E_{GL} << 1$, then the GL theory is self-consistent, otherwise, the GL theory breaks down and we enter the critical fluctuation regime.

1. At $d > 4$, $E_{GL} \sim |t|^{-\frac{d}{2} - 2} << 1$ as $t \to 0$. Landau-Ginsburg theory are qualitatively correct.

2. At $d < 4$, $E_{GL} \sim |t|^{-\frac{d}{2} - 2} >> 1$ as $t \to 0$. The mean-field theory breaks down at $E_{GL} = 1$, i.e. $|t|^{-\frac{d}{2} + 2} = \frac{k_0}{4 \Delta C}$.

   i.e. at $|t| < |t_c| = \left( \frac{k_0}{4 \Delta C} \right)^{2 - \frac{d}{2}}$, we enter the critical region.

3. $d = 4$ is the marginal case.

There exists a **upper critical dimension $d_c = 4$** for the above analysis, such that at $d > d_c$, the quartic term is not important for the critical phenomena. Of course, we need quartic term to spontaneously break the symmetry!
In the above reasoning, we have use the mean field values of critical exponents $\beta = 1/2$, $\gamma = 1$, $\nu = 1/2$. However, for certain mean field transitions whose $r$, $\beta$, $\nu$ have different values, we need modify as follows:

$$\int d^d r \ G(r) \sim k_0 T \chi \sim |t|^{-\sigma}$$

$$\int d^d \bar{m}^2 \sim \int d^d |t|^{2\beta} \sim |t|^{2\beta - \nu d}$$

Suppressing numerical coefficient, we need $|t|^{-\sigma} \ll |t|^{2\beta - \nu d}$ to justify GL mean field theory: $-\sigma > 2\beta - \nu d$

$$\Rightarrow d > \frac{2\beta + \sigma}{\nu} \triangleq d_c$$