Assignment #3  
PHSY 217  Spring 2014

Problem 1. (Goldenfeld Exercise 9-1)
Solution:
(a) Notice that the six critical exponents are not independent since there are four scaling laws. The origin of the scaling laws is that all of the critical exponents can be expressed in terms of $\gamma_+ $ and $\gamma_-$ Hence our strategy would be to rewrite $\delta$ and $\eta$ in terms of $\gamma_+ $ and $\gamma_-$, and then derive the relation.

$\delta$ is defined to be $m - H(V, T = T_c)$. Notice that $m = \langle n \rangle = \frac{\text{tr}(\hat{n} e^{-\beta H})}{\text{tr}(e^{-\beta H})}$, so that any constant term in the action canceled in numerator and denominator.
Thus the scaling of the corresponding singular part of free energy is just
$f_s(t, h) = t^{-d} \tilde{f}_s(t^\gamma_+, h t^\gamma_-)$
which is different from what is investigated in exercise 12-3 (d) where there is additional term that contribute to $f_s$ due to the fact that in total free energy:
$f = -k_b T \ln Z$ and there will be no such cancelling.

Taking derivative with respect to $h$,
$m(t, h) = \tilde{f}_{-d+y_h} m(t t^\gamma_+, h t^\gamma_-)$

Choose $\tilde{t}$ such that the system is driven to $h t^\gamma_- = 1$. Then $t = h^{-\frac{1}{\gamma_-}}$
$m(t, h) = h^{\frac{-d-y_h}{\gamma_+}} m(t h^{-\frac{y_h}{\gamma_+}}, 1)$

Set $t = 0$,
$m(0, h) = h^{\frac{-d-y_h}{\gamma_+}} m(0, 1)$

Hence

$\frac{\delta}{\delta} = \frac{d-y_h}{y_h}$

i.e.

$\delta = \frac{y_h}{d-y_h}$

Then let's derive the expressions for $\nu$ and $\eta$. Let's work in momentum space, and derive the scaling form of $G$. By definition,

$G(\vec{r}, \vec{r'}) = G(\vec{r} - \vec{r'}) := \langle \phi(\vec{r}) \phi(\vec{r'}) \rangle$

$= \int \frac{d^d k}{(2\pi)^d} \frac{d^d k'}{(2\pi)^d} \langle \hat{\phi}_k \hat{\phi}_{k'} \rangle e^{i \vec{k} \cdot \vec{r}} e^{-i \vec{k'} \cdot \vec{r'}}$

$= \int \frac{d^d k}{(2\pi)^d} \frac{d^d k'}{(2\pi)^d} \tilde{G}(k) \delta^d (\vec{k} - \vec{k'}) e^{i \vec{k} \cdot \vec{r}} e^{-i \vec{k'} \cdot \vec{r'}}$

$= \int \frac{d^d k}{(2\pi)^d} \tilde{G}(k) e^{i \vec{k} \cdot (\vec{r} - \vec{r'})}$
Hence \( G(\vec{r}, \vec{r'}) = \int \frac{d^d \vec{k}}{(2\pi)^d} \frac{\langle \phi_{\vec{k}} \phi_{-\vec{k}}^* \rangle}{\det(\Delta)} e^{i \vec{k} \cdot (\vec{r} - \vec{r'})} \)

\[= \frac{1}{V_0} \int \frac{d^d \vec{k}}{(2\pi)^d} \langle \phi_{\vec{k}} \phi_{\vec{k}}^* \rangle e^{i \vec{k} \cdot \vec{r} - i \vec{k} \cdot \vec{r'}} \]

Then under RG,

\[ G(\vec{r}, t, h) = \frac{1}{V} \int \frac{d^d \vec{k}}{(2\pi)^d} \langle \phi_{\vec{k}} \phi_{\vec{k}}^* \rangle e^{-2\omega^2 \Delta t} e^{i \vec{k} \cdot \vec{r} - i \vec{k} \cdot \vec{r'}} \]

\[= \frac{1}{V} \frac{1}{V'} \int \frac{d^d \vec{k}}{(2\pi)^d} \langle \phi_{\vec{k}} \phi_{\vec{k}}^* \rangle e^{-2\omega^2 \Delta t} e^{i \vec{k} \cdot \vec{r} - i \vec{k} \cdot \vec{r'}} \]

\[= \frac{1}{V} \frac{1}{V'} \int \frac{d^d \vec{k}}{(2\pi)^d} \langle \phi_{\vec{k}} \phi_{\vec{k}}^* \rangle e^{i \vec{k} \cdot \vec{r} - i \vec{k} \cdot \vec{r'}} \]

\[= \frac{1}{V} \frac{1}{V'} \int \frac{d^d \vec{k}}{(2\pi)^d} \langle \phi_{\vec{k}} \phi_{\vec{k}}^* \rangle e^{i \vec{k} \cdot \vec{r} - i \vec{k} \cdot \vec{r'}} \]

where we've employed scaling of \( \phi, k \) and Volume as

\[\begin{align*}
\phi_{\vec{k}} &= \phi_{\vec{k}'} \\
\phi_{\vec{k}'} &= \phi_{\vec{k}} e^{i \vec{k} \cdot \vec{r}} \\
V &= V' d^{-d} V
\end{align*}\]

But the scaling of \( \phi \) is related to that of \( H \) as \( y_h = -\frac{1}{2} \omega^2 \Delta t \), as can be easily seen from \( H \int d^d \phi \phi \phi = H \phi (\vec{r} = 0) \) in the Hamiltonian.

So \( G(\vec{r}, t, h) = e^{2(y_h - \Delta t)} G(\vec{r}, t, h) = e^{2(y_h - \Delta t)} G(\vec{r}, t + t \Delta t, h + y_h) \)

Actually the above equality is only in the sense of singular part of \( G \). This is because when we do \( G(\vec{r}, t, h) \) with respect to \( G(\vec{r}, t, h) \), modes lying in \( \{M, N\} \) are lost. But they don't contribute to any singular behaviour anyway as long as \( N \Delta t \gg V' \).

Choose \( t \) such that \( t \Delta t = 1 \), we have

\[ G(\vec{r}, t, h) = t^{2(y_h - y_t)} G(\vec{r}, t + y_t, h - y_t) \]

Write

\[ G(\vec{r} + y_t, t, h) = (t + y_t)^{-2(y_h - y_t)} F_G(\vec{r} + y_t, h + y_t) \]

Then

\[ G(\vec{r}, t, h) = \frac{1}{t^{2(y_h - y_t)}} F_G(\vec{r} + y_t, h + y_t) \]

which recovers scaling hypothesis.

Thus \( 2(y_h - y_t) = d + \eta \), \( V = \frac{1}{y_t} \).

Combining \( \delta = \frac{y_h}{d - y_h} \) together, it is straightforward to get

\[ \delta = \frac{d + \eta}{d - y_h} \]

by cancelling \( y_h \).
(b) Again using the same trick to make $|t| = 1$, we have
\[ f_s(t, h) = t^{-d} f_s \left( t, h, \frac{h}{\|y\|} \right) \]
\[ = t^{-d} f_s \left( t^d \frac{t^d}{\|y\|}, h, \frac{h}{\|y\|} \right) \quad (|t| t^d = 1 \Rightarrow t = |t|^{-\frac{1}{d}}) \]
\[ = |t|^{-d} f_s \left( \pm 1, h t^{1 - \frac{1}{d}} \frac{h}{\|y\|} \right) \]
Plugging in $t = \sqrt{\|y\|}$, we have
\[ f_s(t, h) = |t|^{\frac{d}{d-1}} f_s \left( h t^{1 - \frac{1}{d}} \frac{h}{\|y\|} \right) \]
\[ = |t|^{\frac{d}{d-1}} f_s \left( \frac{h}{\|y\|} \right) \]
where
\[ F^F_\|y\| \left( \frac{h}{\|y\|} \right) = f_s \left( \pm 1, h t^{1 - \frac{1}{d}} \frac{h}{\|y\|} \right) \]
Let's write
\[ F^F_\|y\| \left( \frac{h}{\|y\|} \right) = \left( \frac{h}{\|y\|} \right)^\lambda \phi_\|y\| \left( \frac{h}{\|y\|} \right) \]
and we choose the power $\lambda$ such that
\[ |t|^{\frac{d}{d-1}} \left( \frac{h}{\|y\|} \right)^\lambda = h^{\lambda} \]
which means that
\[ d \sqrt{\|y\|} = 0 \Rightarrow \lambda = \frac{d}{\|y\|} \]
So
\[ f_s(t, h) = h^{\frac{d}{\|y\|}} \phi_\|y\| \left( \frac{h}{\|y\|} \right) \]
such that all dependence on $t$ of $f_s$ is absorbed into the function $\phi_\|y\|$.

Now
\[ f_s(t, h) = \begin{cases} h^{\frac{d}{\|y\|}} \phi_\|y\| \left( \frac{h}{\|y\|} \right), & t > 0, \\ h^{\frac{d}{\|y\|}} \phi_\|y\| \left( \frac{h}{\|y\|} \right), & t < 0 \end{cases} \]
and $f_s$ should be analytic at $t = 0$ for any fixed $h > 0$.

This means that $\phi_\|y\| \left( \frac{h}{\|y\|} \right)$ is the analytic continuation of $\phi_+ \left( \frac{h}{\|y\|} \right)$ to the region $t < 0$ ($t > 0$). If we do analytic continuation of $\phi_+ (\phi_-)$ to $t < 0$ ($t > 0$), so that both two functions are well-defined inside $t \in (-\varepsilon, \varepsilon)$, $\varepsilon > 0$, then clearly
\[ \phi_+ \left( \frac{h}{\|y\|} \right) = \phi_- \left( \frac{h}{\|y\|} \right), \quad t \in (-\varepsilon, \varepsilon). \]

Notice that $\phi_+ \left( \frac{h}{\|y\|} \right)$ and $\phi_- \left( \frac{h}{\|y\|} \right)$ are smooth functions of $t$, but $\phi_+(x)$ and $\phi_-(x)$ themselves may not be smooth with respect to $x$.

Let's write
\[ \phi_+ \left( \frac{h}{\|y\|} \right) = \phi_+ \left( \frac{t^d}{\|y\|} \right)^{-\nu y} = \phi_+ \left( \frac{t}{\|y\|} \right) \]
\[ \phi_- \left( \frac{h}{\|y\|} \right) = \phi_- \left( \frac{t^d}{\|y\|} \right)^{-\nu y} = \phi_- \left( \frac{t}{\|y\|} \right) \]
Now there exists $\varepsilon_1 \geq 0$ and $\varepsilon_2 > 0$, and $h > 0$, s.t.
\[ F^F_\|y\| \left( \frac{t}{\|y\|} \right) = \phi_+ \left( \frac{t}{\|y\|} \right) \]
is smooth in $(-\varepsilon_1, \varepsilon_1) \times (h_0 - \varepsilon_2, h_0 + \varepsilon_2)$, since $F^F_\|y\|$ is analytic at $(t = 0, h = h_0)$.
That $F_{\pm}(\pm \frac{t}{h \omega y_h})$ are analytic in $(-\varepsilon, \varepsilon) \times (h_0 - \varepsilon, h_0 + \varepsilon)$ clearly indicates that $F_{\pm}(x)$ are analytic in $(-\frac{\varepsilon_1}{(2h)(\omega y_h)}, \frac{\varepsilon_1}{(2h)(\omega y_h)})$.

Hence we can expand $F_{\pm}(x)$ in vicinity of $x=0$ as a power series:

$$F_{\pm}(x) = \sum_{n=0}^{\infty} a_n^\pm x^n,$$

for small enough $x$.

Then we must have from $F_{\pm}(\frac{\pm t}{h \omega y_h}) = F_{\pm}(\frac{-t}{h \omega y_h})$ the following

$$\sum_{n=0}^{\infty} a_n^+ \left(\frac{t}{h \omega y_h}\right)^n = \sum_{n=0}^{\infty} a_n^- \left(-\frac{t}{h \omega y_h}\right)^n,$$

i.e.

$$a_n^+ h^{-\frac{n}{\nu y_h}} = (-1)^n a_n^- h^{-\frac{n}{\nu y_h}},$$

which holds for a small range of $h$ around $h_0$. But since both the functions $h^{-\frac{n}{\nu y_h}}$ and $h^{-\frac{n}{\nu y_h}}$ are analytic at $h_0$, we can establish the fact that

$$a_n^+ = (-1)^n a_n^-,$$

$h^{-\frac{n}{\nu y_h}} = h^{-\frac{n}{\nu y_h}}$, for $h$ around $h_0$.

Clearly, we must have $\nu = \nu'$ to satisfy this.
Problem 2. (Goldenfeld Exercise 9-3)

Solution:
(a) As we've been familiar with, the 1D Ising model can be readily solved by the transfer matrix method:

\[ Z_N = \text{tr} e^{H} = \text{tr} (T^N) \]

where

\[ T = \begin{pmatrix} e^{h+K+K_0} & e^{-K+K_0} \\ e^{-K+K_0} & e^{h+K+K_0} \end{pmatrix} \]

Let's then do a partial trace over the even numbered sites:

\[ Z_N = \sum_{s_1 \cdots s_N} \langle s_1 T_1 s_2 \rangle \langle s_2 T_2 s_3 \rangle \cdots \langle s_N T_N s_1 \rangle \]

\[ = \sum_{s_1, s_3, \cdots} \sum_{s_2, s_4, \cdots} \langle s_1 T_1 s_2 \rangle \langle s_2 T_2 s_3 \rangle \cdots \langle s_N T_N s_1 \rangle \]

\[ = \sum_{s_1, s_3, \cdots} \langle s_1 T_1^2 s_2 \rangle \langle s_2 T_2 T_0 s_1 \rangle \cdots \]

\[ = \text{tr} \left( T^2 \left( \frac{N}{2} \right) \right) \]

But

\[ T^2 = \begin{pmatrix} e^{h+2K_0} \cosh (h+2K) & e^{2K_0} \cosh (h) \\ e^{2K_0} \cosh (h) & e^{-h+2K_0} \cosh (-h+2K) \end{pmatrix} \]

and so if we impose

\[ T^2 = \begin{pmatrix} e^{h+K+K'} & e^{-K'+K_0'} \\ e^{-K'+K_0'} & e^{h-K'+K'} \end{pmatrix} \]

we must have

\[ \begin{align*}
& e^{h+2K_0} \cosh (2K+h) = e^{h+K+K'} & - (\oplus) \\
& e^{-h+2K_0} \cosh (2K-h) = e^{-K'+K+K'} & - (\ominus) \\
& e^{2K_0} \cosh (h) = e^{-K'+K_0'} & - (\ominus) 
\end{align*} \]

Then

\[ \begin{align*}
\oplus & \Rightarrow e^{2h'} = e^{2h} \frac{\cosh (2K+h)}{\cosh (2K-h)} \\
\ominus \oplus & \Rightarrow e^{2K' + 2K_0} = e^{4K_0} \cos (2K+h) \cos (2K-h) & - (\oplus) \\
\otimes \ominus & \Rightarrow e^{4K'} = e^{2K_0} \cos (2K+h) \cos (2K-h) \cosh^2 (h) \\
\otimes \ominus & \Rightarrow e^{4K'} = \frac{\cos (2K+h) \cos (2K-h)}{\cosh^2 (h)} 
\end{align*} \]
In summary,

\[
\begin{align*}
\Theta_{2h} &= e^{2h} \frac{\cosh(2k+h)}{\cosh(2k-h)} \\
\Theta_{4k} &= \frac{\cosh(2k+h)}{\cosh(2k-h)} \cos(2k+h) \\
\Theta_{4k_0} &= e^{2k_0} \cos(2k+h) \cos(2k-h) \cosh^2(h)
\end{align*}
\]

The Hamiltonian of 1D Ising model has a time reversal symmetry

\[ H(h, k, (\pm \frac{1}{2})_j) = H(-h, k, (\pm \frac{1}{2})_j) \]

This implies

\[ Z(-h, k, k_0) = Z(h, k, k_0) \]

It is easy to check that under RG operation \( R \), which induces a map on parameter space as defined by solution of the above recursion relation, we have

\[(h', k', k_0') = R(h, k, k_0) \Rightarrow (-h', k', k_0') = R(-h, k, k_0),\]

i.e., \( T \) is time reversal operation

\[
\begin{array}{ccc}
(h, k, k_0) & R & (h', k', k_0') \\
T & & T \\
(-h, k, k_0) & R & (h, k, k_0)'
\end{array}
\]

which means that on parameter space \( R \) and \( T \) commute. So \( R \) preserves time reversal symmetry if there is one in the original problem.

Notice that we've included \( K_0 \) in \( H \) which does not show up in the usual Ising Hamiltonian.

This is because RG will generate constant terms. So even if we start with a model with \( K_0 = 0 \), non zero \( K_0 \) will appear after we do RG. Hence we should include such a term in the flow.

(b) Let's set \( h = 0 \). Then it is easy to read from the second equation of the RG flow equations that

\[ \Theta_{4k} = \cosh^2(2k). \]

To find fixed points, let

\[ \Theta_{4k_0} = \cosh^2(2k_0). \]

Solutions are clearly \( K_0 = 0 \) \& \( +\infty \). The corresponding \( W = e^{-2K} = 0, \infty \).

Since \( K \sim \frac{1}{T} \), \( K_0 = 0 \) corresponds to \( T = +\infty \), the disordered phase; while

\[ \nabla K_0 = +\infty \] corresponds to \( T = 0 \), the critical point.

Flow in \( W \) is

\[ 0 \quad \rightarrow \quad 1 \]
(c) Let's work in vicinity of $T=0$. Here $K$ is very large, so 
\[ \cosh^2(2K) = \left( \frac{e^{2K} + e^{-2K}}{2} \right)^2 \approx \left( \frac{e^{2K}}{2} \right)^2 = \frac{1}{4} e^{4K}. \]

Hence
\[ e^{4K'} = \frac{1}{4} e^{4K} \]
\[ \Rightarrow K' = \frac{1}{4} (4K - 2 \ln 2) = K - \frac{1}{2} \ln 2. \]

This gives
\[ \frac{T}{T'} = \frac{1}{1 - \frac{1}{2} \ln 2} \approx T + \frac{1}{2} (\ln 2) T^2. \]

If we linearize the equation, we will get
\[ 2T\left|_{T=0} \right. = 1, \]
which means that $2^{y_T} = 1 \Rightarrow y_T = 0$ (notice $\lambda = 2$ in this case).

This does not mean that $T$ doesn't flow. $T$ does not change up to linear order, but it flows to high temperature region to higher orders. The flow around $T=0$ is rather slow.

(d) Let's solve the following for $h > 0$:
\[
\begin{align*}
\frac{e^{4K'}}{\cosh(2K+h) \cosh(2K-h)} &= e^{4h} \quad (1) \\
\frac{e^{2h}}{\cosh(2K+h)/\cosh(2K-h)} &= \frac{1}{4} e^{2K+h} = e^{2h} \quad (2)
\end{align*}
\]

If $K = 0$, we see that the equations are satisfied for arbitrary $h$, and $(K' = 0, h' = h)$. So $(K = 0, h$ arbitrary) form a line of critical points.

If $K = \pm \infty$, first equation is satisfied. For second one,
\[ \frac{\cosh(2K+h)}{\cosh(2K-h)} \approx \frac{\frac{1}{2} e^{2K+h}}{\frac{1}{2} e^{2K-h}} = e^{2h}, \]

so that $e^{2h} = e^{4h}$, i.e., $h_T = 0$. Hence $(K = \pm \infty, h = 0)$ is a fixed point.

These are the only fixed points of the RG recursion relation. Notice that for very large $K$, $h' = 2h$, so clearly $y_T = 1$.

The flow diagram is
\[ \begin{array}{c}
h \\
\text{line of fixed points}
\end{array} \]

\[ \begin{array}{c}
\text{(We've only drawn } h > 0 \text{ due to time reversal symmetry)}
\end{array} \]
The results show that except $T=0$, all finite temperatures flow to high temperature region, hence corresponding to disordered phase. So there is no phase transition at a finite temperature.

We can derive the formula for correlation by exact calculation using transfer matrix. (see eqn. 3.111 in Goldfeld’s book or Professor Wu’s lecture notes)

$$\frac{g}{h} = \ln \tanh K.$$

But

$$\ln \tanh K = \ln \frac{e^K + e^{-K}}{e^K - e^{-K}} = \ln \left( \frac{e^K + e^{-K}}{1 - e^{-2K}} \right)$$

$$\ln (1 + e^{-2K})^2 \approx \ln (1 + e^{-2K}) = e^{-K} \ (K \rightarrow +\infty)$$

Hence $g \sim e^{4K} = e^{\frac{4T}{T}} \ (T \rightarrow 0)$, inverse

So $g$ diverges much quicker than any power law behaviour ($\frac{1}{T}$).

This implies $V = +\infty$.

But we’ve seen in problem 9.1 that $V = \frac{1}{y^*_t}$, hence $y^*_t = 0$, in consistent with the result from RG method.

To get $y_t$, we investigate the scaling behaviour of $\frac{g}{h}$ with respect to $h$.

From eqn. 3.109 Goldfeld’s book,

$$\frac{g}{h} = \frac{1}{\log (\lambda_2/\lambda_1)}$$

where

$$\lambda_{1,2} = e^{K} \left[ \cosh h \pm \sqrt{\sinh^2 h + e^{-4K}} \right]$$

Hence $\frac{\lambda_1}{\lambda_2} \sim e^{2h} \ (h \rightarrow 0)$

Thus

$$\log (\lambda_1/\lambda_2) \sim 2h \ (h \rightarrow 0, K \rightarrow +\infty)$$

So

$$g \sim \frac{1}{h}.$$ 

Clearly $y_t = 1$, as consistent with RG result.
Problem 3. (Goldenfeld Exercise 12-3)

Solution:

(a) These two RG flow equations have already been derived in Professor Congjun Wu's lecture notes. The constants $A$ and $B$ take different values for different symmetries. They are all positive anyway, since critical value of $r$ will be shifted downward with respect to its mean-field value, and $u$ has a physical fixed point value for $\varepsilon > 0$.

(b) This part of the problem has also been discussed in Professor Wu's lectures. There are two fixed points, one is Gaussian fixed point with $r^* = u^* = 0$, while the other is Wilson-Fisher fixed point with $(r^* = -\frac{1}{A} \frac{A}{B}, u^* = \frac{\epsilon}{2})$.

If we linearize the RG equation around fixed point, we will have

$$
\begin{bmatrix}
\frac{d \delta r}{ds} \\
\frac{d \delta u}{ds}
\end{bmatrix}
= 
\begin{bmatrix}
\frac{2}{\delta r} \left( \frac{du}{ds} \right) & \frac{\delta u}{\delta u} \left( \frac{du}{ds} \right) \\
\frac{\delta u}{\delta r} \left( \frac{du}{ds} \right) & \frac{\delta u}{\delta u} \left( \frac{du}{ds} \right)
\end{bmatrix}
\begin{bmatrix}
\delta r \\
\delta u
\end{bmatrix}
$$

$$
= 
\begin{bmatrix}
2 - \frac{A u^*}{\epsilon(1+\epsilon)} & A \\
2B & \frac{\epsilon}{1+\epsilon}
\end{bmatrix}
\begin{bmatrix}
\delta r \\
\delta u
\end{bmatrix}
$$

We can then get the familiar results

$$
\begin{bmatrix}
\frac{d \delta r}{ds} \\
\frac{d \delta u}{ds, \text{ Gaussian}}
\end{bmatrix}
= \begin{bmatrix}
2 & A \\
0 & \epsilon
\end{bmatrix} + O(\epsilon^2)
\begin{bmatrix}
\delta r \\
\delta u
\end{bmatrix}, \text{ Gaussian}
$$

$$
\begin{bmatrix}
\frac{d \delta r}{ds} \\
\frac{d \delta u}{ds, \text{ W-F}}
\end{bmatrix}
= \begin{bmatrix}
2 - \frac{A \epsilon}{B} & A \left( 1 + \frac{A}{B} \frac{\epsilon}{2} \right) \\
0 & -\epsilon
\end{bmatrix} + O(\epsilon^2)
\begin{bmatrix}
\delta r \\
\delta u
\end{bmatrix}, \text{ W-F}
$$

The flows are

$E < 0$

$E > 0$
The temperature-like scaling quantity must be zero along the critical line. Hence 
\[ t = t + \frac{1}{2} A u \] is just such a quantity, and actually it is just the first 
two terms of the temperature derived in Professor Wu's lecture notes.

(c) We need the flow equation for the above defined \( t \). We set \( e = 0 \).

\[ \frac{dt}{ds} = \frac{dA}{ds} + \frac{1}{2} A \frac{dA}{ds} \]
\[ = 2t + \frac{A u}{1 + t} - \frac{1}{2} A u^2 \]

At \( d = 4 \), coupling \( u \) is asymptotic free in the infrared, so we only keep terms linear in 
\( u \), since remember that we are lowering the cutoff and \( u \) actually decays.

Then 
\[ \frac{dt}{ds} = 2t + A u - A u \]

But notice that
\[ (2 - A u) t = (2 - A u) (t + \frac{A u}{2}) \]
\[ = 2t + A u - A u - \frac{1}{2} A u^2, \]

We see that up to \( \mathcal{O}(u) \),
\[ \frac{dt}{ds} = (2 - A u) t. \]

This equation is very easy to solve:
\[ \ln \frac{t}{t_0} = \int_0^s (2 - A u) ds' \]
i.e.
\[ t(s) = t_0 e^{\int_0^s (2 - A u(s')) ds'} \]

To get the form of \( t(s) \), we need that of \( u(s) \), which is easy:
\[ \frac{du}{ds} = -B u^2 \]
\[ \Rightarrow \]
\[ \frac{1}{u} - \frac{1}{u_0} = B s \]
\[ \Rightarrow \]
\[ u(s) = \frac{u_0}{1 + B u_0 s} = \frac{u_0}{1 + u_0 B s} \]

Thus
\[ t(s) = t_0 e^{\int_0^s (2 - \frac{A u}{1 + u_0 B s}) ds} \]
\[ = t_0 e^{2 s} \cdot e^{-\int_0^s \frac{A u ds'}{1 + u_0 B s'}} \]
in which
\[ \int_0^s \frac{A u ds'}{1 + u_0 B s'} = \frac{A}{B} \ln (1 + u_0 B s) \sim \frac{A}{B} \ln (u_0 B s) \] \text{if} \( s \gg 1 \).

In summary,
\[ t(s) = t_0 (u_0 B)^{-\frac{A}{B}} e^{2 s} \cdot s^{-\frac{A}{B}} \sim t_0 t^2 (\log t)^{-\frac{A}{B}} \] (since \( s = \ln t \))
To get relation between $\ell$ and $t$, consider we drive the system to $|t(s)|=1$, and write $t_0 = t$. Then

$$\ell^t \sim \frac{1}{\ell^t} (\log \ell)^{\frac{A}{B}}$$

Hence

$$\ell^t \sim \frac{1}{\ell^t} (\log \ell)^{\frac{A}{B}}$$

$$\Rightarrow \ell \sim \ell^{-1} (\log \ell)^{\frac{A}{B}}$$

If we are only interested in leading scaling behaviour of $\ell$, we can set $\ell = \ell^{-1} - \frac{1}{2}$ in $\log \ell$, so that

$$\ell \sim \ell^{-1} \left( -\frac{1}{2} \log \ell + 1 \right)^{\frac{A}{B}}$$

Dropping irrelevant constants, we get

$$\ell \sim \ell^{-1} (-\log \ell)^{\frac{A}{B}}$$

For Ising universality class, $\frac{A}{B} = \frac{1}{2}$, and for $t > 0$,

$$\ell \sim t^{-1/2} (-\log +)^{\frac{1}{2}}$$

Relation between $\frac{1}{2}$ and $\ell$ is straightforward:

$$\frac{1}{2}(t_0) = \ell \frac{1}{2}(t(s))$$

Again set $t_0 = t$ and $|t(s)|=1$, we see

$$\frac{1}{2}(t) = \ell \frac{1}{2}(t(s)) \sim t$$

So

$$\frac{1}{2} \sim t^{-1/2} (-\log +)^{\frac{1}{2}} \quad (t > 0)$$

(d) From a naive scaling form of free energy density that we've been familiar with we get

$$f_s(t, u) = t^{-d} f_s(t, u)$$

If $t$ is very large, $u$ approaches $u_x$, so that

$$f_s(t, u) = t^{-d} f_s(t, u_x)$$

We've seen in part (c) that if we scale $t$ to $t = \pm$, the scaling factor that we need is $\ell \sim t^{-1/2} (-\log +)^{\frac{1}{2}} \sim t^{-1/2} (-\log +)^{\frac{1}{2}} \quad (t > 0)$.

Plugging this into $f$ we get (set $d = 4$)

$$f_s(t, u) \sim t^{2/3} (-\log +)^{-2/3}$$

Specific heat $\sim \frac{d}{dt} f_s \sim (-\log +)^{-2/3}$ to leading order singularity.

But this approach may in principle be wrong, and actually it is. The reason is that while we get the flow of field theories in coupling constant space, we've dropped a lot of constant terms in the action, for example, Gaussian integral of free part of the fast modes, and change of functional measure when we rescale the field $\phi$.

These constants actually do not appear and cancel between numerator and denominator.
if we calculate correlation functions. But they in principle should be kept if we are interested in free energy. When we do RG many times (i.e. when \( t \) is very large), the accumulation of these terms may contribute to singular behaviours which has to be taken into account in \( t_s \).