Lect 2. path integral representation of spin

§1. Schwinger boson representation

$$S_x = [a_1^\dagger a_2 + a_2^\dagger a_1]/2, \quad S_y = [a_1^\dagger a_2 - a_2^\dagger a_1]/2i, \quad S_z = (a_1^\dagger a_1 - a_2^\dagger a_2)/2$$

under the constraint of $S = (a_1^\dagger a_1 + a_2^\dagger a_2)/2$.

The spin state $|S,m\rangle = \dfrac{(a_1^\dagger)^{S+m}}{\sqrt{(S+m)!}} \dfrac{(a_2^\dagger)^{S-m}}{\sqrt{(S-m)!}} |0\rangle$.

The Euler angle representation of SU(2) Rotation

$$R(\phi, \theta, \chi) = e^{-i\phi S_z} e^{-i\theta S_y} e^{-i\chi S_z},$$

The transformation of Schwinger bosons

$$a_1'^\dagger = R a_1^\dagger R^{-1}, \quad a_2'^\dagger = R a_2^\dagger R^{-1}$$

Let us first calculate $e^{-i\alpha S_z} \begin{pmatrix} a_1^\dagger \\ a_2^\dagger \end{pmatrix} e^{i\alpha S_z} \triangleq f(\alpha)$

$$i \frac{d}{d\alpha} f(\alpha) = e^{-i\alpha S_z} \begin{pmatrix} [S_z, a_1^\dagger] \\ [S_z, a_2^\dagger] \end{pmatrix} e^{i\alpha S_z} = \frac{1}{2} \begin{pmatrix} e^{-i\alpha S_z} a_1^\dagger e^{+i\alpha S_z} \\ -e^{-i\alpha S_z} a_2^\dagger e^{+i\alpha S_z} \end{pmatrix}$$

$$\Rightarrow e^{-i\alpha S_z} \begin{pmatrix} a_1^\dagger \\ a_2^\dagger \end{pmatrix} e^{i\alpha S_z} = \begin{pmatrix} a_1^\dagger e^{-\frac{i}{2}\alpha} \\ a_2^\dagger e^{\frac{i}{2}\alpha} \end{pmatrix} = \begin{pmatrix} e^{-\frac{i}{2}\alpha} & 0 \\ 0 & e^{\frac{i}{2}\alpha} \end{pmatrix} \begin{pmatrix} a_1^\dagger \\ a_2^\dagger \end{pmatrix}$$

Next, we calculate $\quad e^{-i\beta S_y}\begin{pmatrix} a_1^+ \\ a_2^+ \end{pmatrix} e^{i\beta S_y}$

$$\frac{d}{d\beta}\left[e^{-i\beta S_y}\begin{pmatrix} a_1^+ \\ a_2^+ \end{pmatrix} e^{i\beta S_y}\right] = e^{-i\beta S_y}\begin{pmatrix} -i[S_y, a_1^+] \\ -i[S_y, a_2^+] \end{pmatrix} e^{i\beta S_y} = e^{-i\beta S_y}\begin{pmatrix} \frac{1}{2}a_2^+ \\ -\frac{1}{2}a_1^+ \end{pmatrix} e^{i\beta S_y}$$

$$e^{-i\beta S_y}\begin{pmatrix} a_1^+ \\ a_2^+ \end{pmatrix} e^{i\beta S_y} = \begin{pmatrix} \cos\frac{\beta}{2} & \sin\frac{\beta}{2} \\ -\sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{pmatrix}\begin{pmatrix} a_1^+ \\ a_2^+ \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} a_1'^+ \\ a_2'^+ \end{pmatrix} = R\begin{pmatrix} a_1^+ \\ a_2^+ \end{pmatrix} R^{-1} = e^{-i\phi S_z} e^{-i\theta S_y} e^{-i\chi S_z}\begin{pmatrix} a_1^+ \\ a_2^+ \end{pmatrix} e^{i\chi S_z} e^{i\theta S_y} e^{i\phi S_z}$$

$$= \begin{pmatrix} e^{-i\frac{\chi}{2}} & 0 \\ 0 & e^{i\frac{\chi}{2}} \end{pmatrix}\begin{pmatrix} \cos\frac{\beta}{2} & \sin\frac{\beta}{2} \\ -\sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{pmatrix}\begin{pmatrix} e^{-i\phi/2} & 0 \\ 0 & e^{+i\phi/2} \end{pmatrix}\begin{pmatrix} a_1^+ \\ a_2^+ \end{pmatrix} = \begin{pmatrix} \cos\frac{\theta}{2}\, e^{-i\frac{\phi+\chi}{2}}, & \sin\frac{\theta}{2}\, e^{i\frac{\phi-\chi}{2}} \\ -\sin\frac{\theta}{2}\, e^{i\frac{-\phi+\chi}{2}}, & \cos\frac{\theta}{2}\, e^{i\frac{\phi+\chi}{2}} \end{pmatrix}\begin{pmatrix} a_1^+ \\ a_2^+ \end{pmatrix}$$

define $\quad u = \cos\frac{\theta}{2}\, e^{-i\phi/2}, \quad v = \sin\frac{\theta}{2}\, e^{i\phi/2}$

$$\begin{pmatrix} a_1'^+ \\ a_2'^+ \end{pmatrix} = \begin{pmatrix} u\, e^{-i\chi/2}, & v\, e^{-i\chi/2} \\ -v^* e^{i\frac{\chi}{2}}, & u^* e^{i\frac{\chi}{2}} \end{pmatrix}\begin{pmatrix} a_1^+ \\ a_2^+ \end{pmatrix}.$$

§2. Spin-coherent state representation

$$|\Omega\rangle = R(\chi, \theta, \phi)\,|S, S\rangle = e^{-iS_z\phi}\, e^{-iS_y\theta}\, e^{-iS_z\chi}\,|S, S\rangle$$

where $\qquad \hat{\Omega} = (\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta)$

$$|\Omega\rangle = \frac{(a_1^{+\prime})^{S+m}}{\sqrt{(S+m)!}}\,|SS\rangle = \left(e^{-i\chi/2}\right)^{2S}\frac{(u a_1^+ + v a_2^+)^{2S}}{\sqrt{2S!}}\,|0\rangle$$

$$= e^{-iS\chi} \sum_m \binom{2S}{S+m} \frac{u^{S+m} v^{S-m}}{\sqrt{2S!}} (a_1^\dagger)^{S+m} (a_2^\dagger)^{S-m} |0\rangle$$

$$= e^{-iS\chi} \sqrt{2S!} \sum_m \frac{u^{S+m} v^{S-m}}{\sqrt{(S+m)!}\sqrt{(S-m)!}} |S,m\rangle$$

Inner product

$$\langle \Omega | \Omega' \rangle = e^{iS(\chi-\chi')} \; 2S! \sum_m \frac{(u^* u')^{S+m} (v^* v')^{S-m}}{(S+m)!\,(S-m)!} = e^{iS(\chi-\chi')} (u^* u' + v^* v')^{2S}$$

$$u^* u' + v^* v' = \cos\frac{\theta}{2} \cos\frac{\theta'}{2} e^{+i\frac{1}{2}(\phi-\phi')} + \sin\frac{\theta}{2} \sin\frac{\theta'}{2} e^{-i\frac{1}{2}(\phi-\phi')}$$

$$= \cos\frac{\theta-\theta'}{2} \cos\frac{\phi-\phi'}{2} + i \cos\frac{\theta+\theta'}{2} \sin\frac{\phi-\phi'}{2}$$

$$|u^* u' + v^* v'|^2 = \cos^2\frac{\theta}{2} \cos^2\frac{\theta'}{2} + \sin^2\frac{\theta}{2} \sin^2\frac{\theta'}{2} + 2 \frac{\sin\theta}{2} \frac{\sin\theta'}{2} \cos(\phi-\phi')$$

$$= \frac{1+\cos\theta}{2} \frac{1+\cos\theta'}{2} + \frac{(1-\cos\theta)(1-\cos\theta')}{2\cdot 2} + \frac{1}{2} \sin\theta \sin\theta' \cos(\phi-\phi')$$

$$= \frac{1}{2}\left[1 + \cos\theta\cos\theta' + \sin\theta\sin\theta' \cos(\phi-\phi') \right] = \frac{1}{2}(1 + \hat{\Omega}\cdot\Omega')$$

$$\Rightarrow \langle \Omega | \Omega' \rangle = \left(\frac{1+\hat{\Omega}\cdot\hat{\Omega}'}{2} \right)^{2S} e^{+iS\psi}, \quad \psi = 2\arctg\left[\tanh\left(\frac{\phi-\phi'}{2}\right) \frac{\cos\frac{\theta+\theta'}{2}}{\cos\frac{\theta-\theta'}{2}} \right] + \chi - \chi'$$

resolution identity

$$\frac{2S+1}{4\pi} \int d\hat{n}\, |\hat{\Omega}\rangle\langle\hat{\Omega}| = \frac{2S+1}{4\pi} (2S!) \int d\hat{n} \sum_m \frac{|u|^{2S+2m} |v|^{2S-2m}}{(S+m)!\,(S-m)!} |Sm\rangle\langle Sm|$$

$$= \int \frac{d\hat{n}}{4\pi} (2S+1)! \sum_m \frac{\left(\frac{1+\cos\theta}{2}\right)^{S+m} \left(\frac{1-\cos\theta}{2}\right)^{S-m}}{(S+m)!\,(S-m)!} |Sm\rangle\langle Sm|$$

$$\int \frac{d\Omega}{4\pi} \left(\frac{1+\cos\theta}{2}\right)^{S+m} \left(\frac{1-\cos\theta}{2}\right)^{S-m} = \frac{1}{2}\int_{-1}^{1} dx \left(\frac{1+x}{2}\right)^{S+m} \left(\frac{1-x}{2}\right)^{S-m}$$

set $y = \frac{x+1}{2}$ $\Rightarrow \frac{1}{2}\int_{-1}^{1} dx \left(\frac{x+1}{2}\right)^{S+m}\left(\frac{1-x}{2}\right)^{S-m} = \int_0^1 dy\; y^{S+m}(1-y)^{S-m}$

$$= \frac{(S+m)!\,(S-m)!}{(2S+1)!}$$

$$\Rightarrow \frac{2S+1}{4\pi}\int d\Omega\, |\hat{\Omega}\rangle\langle\hat{\Omega}| = \sum_m |Sm\rangle\langle Sm| = 1$$

Exercise: $\frac{(S+1)(2S+1)}{4\pi}\int d\Omega\; \hat{\Omega}^\alpha\, |\hat{\Omega}\rangle\langle\hat{\Omega}| = S^\alpha, \quad \alpha = \hat{x}, \hat{y}, \hat{z}.$

§ path integral representation of partition function

$$\mathcal{Z} = \mathrm{Tr}\left[e^{-\beta \hat{H}}\right] \quad \text{or} \quad \mathrm{Tr}\left[T_\tau\left\{\exp\int_0^\beta d\tau\,(-\hat{H}(\tau))\right\}\right]$$

$$= \lim_{N_\epsilon \to +\infty} \mathrm{Tr} \prod_{n=0}^{N_\epsilon - 1} \left[1 - \epsilon \hat{H}(\tau_n)\right]$$

Insert resolution - identity

$$\mathcal{Z} = \lim_{N_\epsilon\to+\infty} \iint \left[\prod_{i,\tau} d\Omega_i(\tau)\right] \prod_{\tau=\epsilon}^{\beta} \langle \Omega(\tau)|\, 1 - \epsilon \hat{H}(\tau_n)\, |\Omega(\tau-\epsilon)\rangle$$

$$= \lim_{N_\epsilon\to} \iint \left[\prod_{i,\tau} d\Omega_i(\tau)\right] \prod_{\tau=\epsilon}^{\beta} \langle \Omega(\tau)|\Omega(\tau-\epsilon)\rangle \left[1 - \epsilon H(\tau)\right],$$ where

$$H(\tau) = \frac{\langle \Omega(\tau)|\, \hat{H}\, |\Omega(\tau-\epsilon)\rangle}{\langle \Omega(\tau)|\Omega(\tau-\epsilon)\rangle}$$

$$\langle \hat{\Omega}(\tau+\epsilon) | \hat{\Omega}(\tau) \rangle \xrightarrow[\text{first order}]{\text{keep to } \epsilon\text{'s}} \exp\left[i S \, \epsilon \left(\sum_i \dot{\phi}_i \cos(\theta_i(\tau)) + \dot{\chi} \right) \right]$$

for $H(\tau) = \dfrac{\langle \Omega(\tau) | \hat{H} | \Omega(\tau-\epsilon) \rangle}{\langle \Omega(\tau) | \Omega(\tau-\epsilon) \rangle}$, we use $\langle \Omega(\tau) | \hat{H} | \Omega(\tau) \rangle$ to approximate by neglecting high order of ϵ.

because $(\hat{\Omega} \cdot \vec{S}) | \hat{\Omega} \rangle = S | \hat{\Omega} \rangle$, by expanding $\vec{S}$ for its components along $\vec{\Omega}$, and transverse directions, the above expression is just the classic expression.

$$\langle \Omega_i(\tau) | \hat{H} | \Omega_i(\tau) \rangle = H(\hat{\Omega}_i).$$

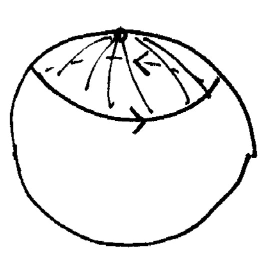
Then we have the partition function

$$Z = \oint D\hat{\Omega}(\tau) \cdot \exp\left[- S(\hat{\Omega}) \right]$$

$$S(\hat{\Omega}) = + i S \sum_i \omega[\hat{\Omega}_i] + \int_0^\beta d\tau \, H[\hat{\Omega}(\tau)]$$

where $D[\hat{\Omega}(\tau)] = \lim_{N_\epsilon \to +\infty} \prod_{i,n} d\hat{\Omega}_i(\tau_n)$, $\quad \omega[\hat{\Omega}_i] = \int_0^\beta d\tau (-\dot{\phi} \cos\theta)$

$$= \oint d\phi \, (1 - \cos\theta)$$

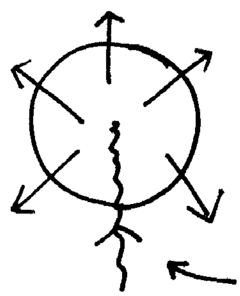


" ω " is the Berry phase part, which corresponds to the area enclosed by the closed path.

$$\omega = \int_0^\beta d\tau \; \vec{A}(\Omega)\,\dot{\hat{\Omega}}, \quad \text{where } (\nabla\times\vec{A})\,\hat{\Omega} = 1,$$

$\vec{A}$ is the vector-potential for a magnetic monopole.

The standard form: $\vec{A} = \dfrac{1-\cos\theta}{\sin\theta}\,\hat{e}_\phi$



Dirac string

§. Propagator

by $\tau \to it$

$$G(\Omega_t, \Omega_0; t) = \langle \Omega_t |\, T\left(\exp \int_0^t dt' \; -i\, H(t')\right) | \Omega_0 \rangle$$

$$= \int_{\Omega_0}^{\Omega_t} D\Omega(t') \, \exp[\, i\, S[\hat{\Omega}]\,], \quad \text{where } S[\hat{\Omega}] \text{ is the}$$

real time action $\quad S[\hat{\Omega}] = \int_0^t dt' \left\{ S\, \Sigma\, A\cdot\dot{\Omega} - H[\Omega(t')] \right\}$

§ Equation of motion and large-S expansion

we will find the saddle point equation

$$\frac{\delta}{\delta\hat{\Omega}} S[\hat{\Omega}] \Big|_{\Omega^{cl,\alpha}} = 0, \quad \text{subject to boundary condition}$$

$$\hat{\Omega}^{cl,\alpha}(0) = \Omega_0, \quad \hat{\Omega}^{cl,\alpha}(t) = \Omega_t$$

$$\delta\left[\int_0^t A\cdot\dot{\Omega}\,dt'\right] = \int_0^t dt' \left[\frac{\partial A^\alpha}{\partial \Omega^\beta}\,\delta\Omega^\beta\,\dot{\Omega}^\alpha + A^\alpha \cdot \frac{d}{dt}\,\delta\hat{\Omega}^\alpha \right]$$

$$+ \left[\frac{\partial A^\alpha}{\partial \Omega^\beta}\,\dot{\Omega}^\beta\,\delta\hat{\Omega}^\alpha - \frac{\partial A^\alpha}{\partial \hat{\Omega}^\beta}\,\dot{\Omega}^\beta\,\delta\Omega^\alpha \right]$$

$$= \int_0^t dt' \; \frac{\partial A^\alpha}{\partial \Omega^\beta}\left[\dot{\Omega}^\alpha\,\delta\Omega^\beta - \dot{\Omega}^\beta\,\delta\Omega^\alpha\right] + \int_0^t dt' \,\frac{d}{dt'}\left(\vec{A}\cdot\delta\hat{\Omega}\right)$$

$$= \int_0^t dt' \left[\delta_{\alpha\alpha'}\delta_{\beta\beta'} - \delta_{\alpha'\beta}\delta_{\alpha\beta'}\right] \frac{\partial A^\alpha}{\partial \Omega^\beta}\left[\dot{\Omega}^{\alpha'}\,\delta\Omega^{\beta'}\right] = \int_0^t dt'\,\epsilon^{\gamma\alpha\beta}\,\epsilon^{\gamma\alpha'\beta'}\,\frac{\partial A^\alpha}{\delta \Omega^\beta}\,\dot{\Omega}^{\alpha'}\,\delta\Omega^{\beta'}$$

$$= \int_0^t dt'\;\hat{\Omega}\cdot\left(\dot{\hat{\Omega}}\times\delta\hat{\Omega}\right) = \int_0^t dt'\;\delta\hat{\Omega}\cdot\left(\hat{\Omega}\times\dot{\hat{\Omega}}\right)$$

$$\Rightarrow\quad \hat{\Omega}\times\dot{\hat{\Omega}} = \frac{\partial H}{\partial\hat{\Omega}}[\hat{\Omega}]$$

$$\hat{\Omega}\times[\hat{\Omega}\times\dot{\hat{\Omega}}] = \hat{\Omega}\times\frac{\partial H}{\partial\hat{\Omega}}[\hat{\Omega}]$$

$$\left[(\hat{\Omega}\cdot\dot{\hat{\Omega}})\hat{\Omega} - (\hat{\Omega}\cdot\hat{\Omega})\dot{\hat{\Omega}}\right] = \boxed{\dot{\hat{\Omega}} = \hat{\Omega}\times\left(-\frac{\partial H}{\partial\hat{\Omega}}\right)}$$

if $H = -\vec{B}\cdot\vec{S}$

$$\Rightarrow\quad \dot{\hat{\Omega}} = \hat{\Omega}\times\hat{B}$$

$$\dot{\Omega}_\alpha = \frac{1}{i\hbar}[\Omega_\alpha \; (-) B_\beta \Omega_\beta]$$

$$= \frac{-1}{i\hbar} B_\beta \; i\hbar\,\epsilon_{\alpha\beta\gamma}\,\Omega_\gamma$$

$$= -\epsilon_{\alpha\beta\gamma}\,B_\beta\,\Omega_\gamma = \left(\vec{\Omega}\times\vec{B}\right)$$