Take home midterm - Phy212b, due Feb. 24 10:am (Monday). Please turn your solutions to Wang Yang at MHA 5206. If you turn in to my or Wang's mailbox, you need to write us an email so that we can confirm.

Please write down a sentence to express that you have done the midterm independently without discussion with others, and sign your name. Another thing to mention is that handwriting is important. A good handwriting is a virtue and will benefit you. Readers and graders will appreciate.

Problem 1. (12 points) D-matrix. Using Mathematica is not forbidden but not recommended either.

Calculate the rotation matrix of the small d-matrix defined as $d^j_{m'm}(\beta) = \langle jm'|e^{-iJ_y\beta}|jm\rangle$ for $j = \frac{3}{2}$.

You are not allowed to quote the general lengthy formula of d-matrix, which is not good for you to understand the d-matrix. You need to present the calculation details by using the Schwinger boson method. Please organize your results in a neat way in the following 4×4 matrix form. And also make your matrix form clean such that we can easily determine its row and column indices. If you cannot get all the 16 elements, leave those you cannot blank.

$$
\begin{pmatrix}
d^j_{\frac{3}{2},\frac{3}{2}}(\beta) & d^j_{\frac{3}{2},\frac{1}{2}}(\beta) & d^j_{\frac{3}{2},-\frac{1}{2}}(\beta) & d^j_{\frac{3}{2},-\frac{3}{2}}(\beta) \\[2mm]
d^j_{\frac{1}{2},\frac{3}{2}}(\beta) & d^j_{\frac{1}{2},\frac{1}{2}}(\beta) & d^j_{\frac{1}{2},-\frac{1}{2}}(\beta) & d^j_{\frac{1}{2},-\frac{3}{2}}(\beta) \\[2mm]
d^j_{-\frac{1}{2},\frac{3}{2}}(\beta) & d^j_{-\frac{1}{2},\frac{1}{2}}(\beta) & d^j_{-\frac{1}{2},-\frac{1}{2}}(\beta) & d^j_{-\frac{1}{2},-\frac{3}{2}}(\beta) \\[2mm]
d^j_{-\frac{3}{2},\frac{3}{2}}(\beta) & d^j_{-\frac{3}{2},\frac{1}{2}}(\beta) & d^j_{-\frac{3}{2},-\frac{1}{2}}(\beta) & d^j_{-\frac{3}{2},-\frac{3}{2}}(\beta)
\end{pmatrix} . \tag{1}
$$

Problem 2 (15 points) Perturbation theory. Planar rotor in the external quadrupolar potential.

Consider a particle with mass M on a planar ring with radius R whose position is denoted by the azimuthal angle ϕ. The un-perturbed Hamiltonian is defined as

$$H_0 = \frac{L_\phi^2}{2I},\tag{2}$$

where $I = MR^2$, and L_ϕ is the angular momentum associated with the angle ϕ.

1) For H_0, what is eigen-energy E_m^0 for the eigenstate with angular momentum quantum number m? What is the eigenfunction Ψ_m^0? Show that except for the ground state $m = 0$, all other states are doubly degenerate for $\pm m$. Do you know what symmetry protects this double degeneracy pattern?

Now imagine that we apply an external quadrupolar potential as

$$H' = -V_0 \cos 2\phi.\tag{3}$$

2) After introducing H', what is the residual symmetry of the system? Do you still expect energy level degeneracies in general? Why? Under what conditions can we treat H' as a perturbation?

(Hint: please pay attention to both reflection and rotation symmetries.)

We next calculate the corrections to wavefunctions and energies for Ψ_m for $m = 0, \pm 1$, respectively.

3) If the particle originally lies in the ground state $m = 0$, what is the new ground state wavefunction (to the 1st order) and the ground state energy (to the 2nd order)? What is the symmetry property of the new ground state?

4) For states with $m = \pm 1$, we need to use degenerate perturbation theory. What are the new energies (to the 2nd order) and new forms of the eigenfunctions (to the 0th order). (If you can work out the 1th order eigen wavefunctions, there are extra credits). What are the symmetry properties of the new eigenstates?

Problem 3. Quantum transition – branching ratio (13 points).

In considering the spin-orbit coupling, the $2p$ energy level of the hydrogen atom splits into a pair of doublets with total angular momenta $j = \frac{3}{2}$ and $\frac{1}{2}$ denoted as $2p_{\frac{3}{2}}$ and $2p_{\frac{1}{2}}$, respectively. Imagine that the initial state of the hydrogen atom is in the $1s$ state with electron spin up, i.e., $j_z = s_z = \frac{1}{2}$. Consider the transition from $1s$ to $2p$, and neglect the small energy difference between $2p_{\frac{3}{2}}$ and $2p_{\frac{1}{2}}$.

1) If the incident light is linearly polarized, and the electric field polarization is $\vec{E} = E_x \hat{e}_x + E_y \hat{e}_y + E_z \hat{e}_z$. Calculate the ratio of the amplitude square of the dipole matrix elements $|\langle 2p_{j,j_z} | \vec{E} \cdot \vec{r} | 1s_{\frac{1}{2},\frac{1}{2}} \rangle|^2$ for all the values of $j = \frac{3}{2}$ and $j_z = -j, ..., j$.

Hint: you need to use the expression of spin-orbit coupled spherical harmonics composed by $Y_{lm}(\theta,\phi)$ and spin-$\frac{1}{2}$ spinor for the special case $l = 0$ and 1. You were asked to derive them last quarter. Please find them and you can use them.

2) Let us consider the total transition probability from $1s_{\frac{1}{2},\frac{1}{2}}$ to all states of $2p_{\frac{3}{2}}$, and also the total transition probability to both states of of $2p_{\frac{1}{2}}$. Show that the ratio between these two probabilities is independent of the polarization direction of linearly polarized incident light. What is the value of this ratio?

3) If the incident light wavevector is along the x-direction and unpolarized, what is the branching ratio for the transitions from $1s_{\frac{1}{2},\frac{1}{2}}$ to $2p_{j=\frac{3}{2},j_z}$ with $j_z = \frac{3}{2}, \frac{1}{2}, -\frac{1}{2}$, and $-\frac{3}{2}$, respectively.