Lecture 5 — Landau levels and quantum Hall effect

$$H_{2D} = \frac{(\vec{P} - \frac{e}{c}\vec{A})^2}{2m}, \quad \text{where} \quad \vec{A} = +B(y, 0), \boxed{\begin{array}{c}\text{only depends on} \\ y, \text{not on } x\end{array}}$$

Then $(\nabla \times \vec{A})_z = \partial_x A_y - \partial_y A_x = -B$; this gauge is first used by Landau

thus is called Landau gauge. It's simpler than the symmetric gauge

$\vec{A} = \frac{-B}{2}\hat{z} \times \vec{r}$, but it doesn't preserve rotation symmetry, explicitly.

$$H_{2D} = \frac{(P_x - \frac{e}{c}By)^2}{2m} + \frac{P_y^2}{2m} = \frac{P_y^2}{2m} + \frac{1}{2}m\omega_c^2\left(y - l_B^2\frac{P_x}{\hbar}\right)^2$$

where $\omega_c = \frac{|e B|}{mc}$, $l_B = \sqrt{\frac{\hbar c}{|eB|}}$.

We solve the wavefunctions: $\psi_{n,k_x}(\vec{r}) = f_n(y)\, e^{ik_x x}$

$\Rightarrow \left[\frac{P_y^2}{2m} + \frac{1}{2}m\omega_c^2(y - l_B^2 k_x)^2\right] f_n = E_n f_n$

$\Rightarrow E_n = \left(n + \frac{1}{2}\right)\hbar\omega_c$, which is independent of k_x.

and $f_n(y) = \phi_n(y - y_0(k_x))$ which is a center-shifted

harmonic oscillator wavefunction

$$\phi_n(x) = \left[\frac{1}{\sqrt{\pi}\, 2^n n!\, l_B}\right]^{1/2} H_n\left(\frac{x}{l_B}\right) e^{-\frac{x^2}{2l_B^2}}, \quad \text{where } H_n \text{ is the}$$

★ Hermite polynormial.

$$H_0(x) = 1, \quad H_1(x) = 2x, \quad H_3(x) = 4x^2 - 2, \cdots$$

a general formula for Hermite polynomial is

$$H_n(x) = (-)^n e^{x^2} \frac{d^n}{dx^n}\left(e^{-x^2}\right)$$

it's generation function is

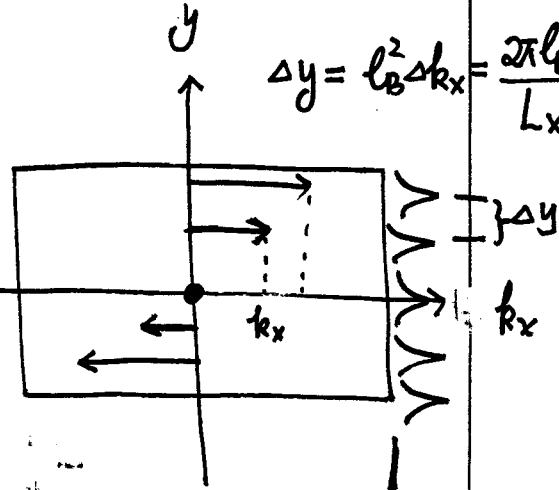$$e^{-s^2 + 2xs} = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} s^n$$

You can look them up in any math-physics ~~test~~ books.

orthnormal condition

$$\int_{-\infty}^{+\infty} H_m(x)\, H_n(x)\, e^{-x^2}\, dx = \sqrt{\pi}\, 2^n\, n!\, \delta_{nm}\,.$$

§ spatial separation of Chiral modes and non-commutative geometry.

$\Delta y = l_B^2 \Delta k_x = \frac{2\pi l_B^2}{L_x}$

The system behaves like a set one-dimensional harmonic oscillators along the y-direction.
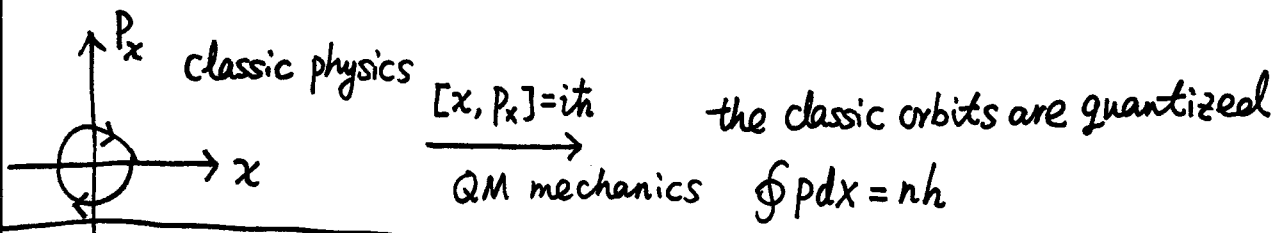
(Let us focus on the LLL with $n = 0$, such that all the oscillators are in the viabrational ground states)

The interesting thing is that the centers of these harmonic modes are correlated with its momentum along the x-direction. For those with $k_x > 0$, their centers are shifted up, and for those with $k_y < 0$, their centers are shifted down.

In other words, the y-axis plays the role of the momentum of the x-axis. If we only keep the LLL states (this is justified in the case of the gap between LLs $\hbar w_c$ is much larger than all the other energy scales that we are interested, i.e. in the limit of $\hbar w_c \to +\infty$. This process is called LLL projection).

$$[x, y]_{LLL} = [x, l_B^2 P_x/\hbar] = i l_B^2 \qquad \longleftarrow \text{non-commutative geometry}$$

classic physics

$[x, P_x] = i\hbar$

QM mechanics

the classic orbits are quantized

$\oint p\, dx = nh$

classic orbits in phase space has chirality!

in usual QM $[x, y] = 0$,

after LLL projection, $[x, y]_{LLL} = i l_B^2$.

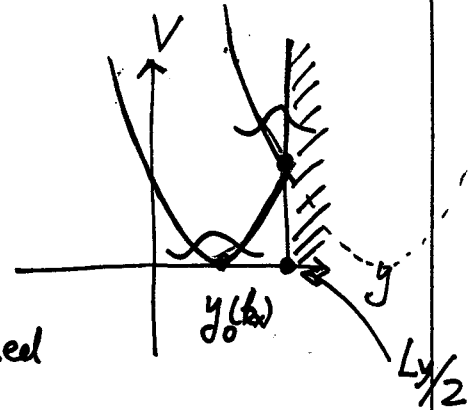Thus the 2D LL in the LLL projection, the (xy)-plane behaves as the phase space of (x, k_x).

✿ Edge spectra:

The effective potential for the state with momentum k_x, is

$$V_{k_x} = \frac{1}{2} m \omega_c^2 (y - \ell_B^2 k_x)^2.$$

If we impose a boundry along the $y = \frac{L_y}{2}$, thus V_{k_x} is truncated to $+\infty$ at $y = \frac{L_y}{2}$.
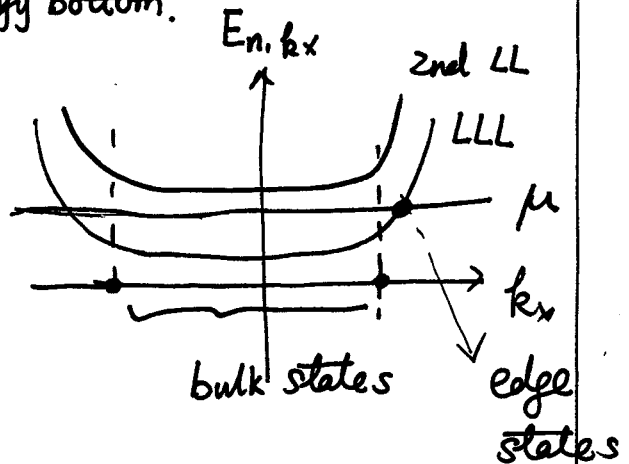
For states $k_x < \frac{L_y}{2\ell_B^2}$, its center of $V_{k_x}(y)$ is away from the boundary, thus it's not affected but the boundary. But as $k_x > \frac{L_y}{2\ell_B^2}$, its bottom is cut by the boundary, and thus its energy is pushed up. Even at the classic level, we have

$$E_n(k_x) = \underbrace{\frac{1}{2} m \omega_c^2 (\ell_B^2)^2 (k_x - \frac{L_y}{2\ell_B^2})^2}_{\text{energy bottom.}} + (n + \frac{1}{2}) \hbar \omega$$

Thus we have the LL spectra with imposing boundaries at $-\frac{L_y}{2}$, and $\frac{L_y}{2}$.

✱ Chirality of the edge modes

The bulk states actually do not carry current. This is also inconsistent with the classical picture — Electrons do cyclotron motion, and thus no charge transport. Now we explicitly verify it.

$$j_x = \frac{1}{2m}\left[\psi^*(P_x - \frac{e}{c}A_x)\psi - \psi(P_x + \frac{e}{c}A_x)\psi^*\right]$$

For bulk state $\quad \psi_{n,k_x}(x,y) = \phi_n(y - y_0(k)) \dfrac{e^{ik_x x}}{\sqrt{L_x}}$

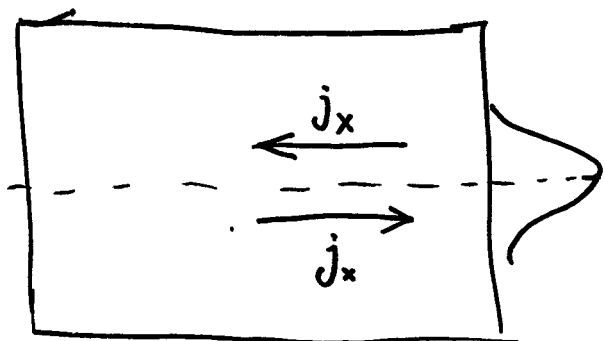$$(P_x - \frac{e}{c}A_x)\psi_{n,k_x} = [\hbar k_x - \frac{eB}{c}y]\,\psi_{n,k_x}$$

$$(P_x + \frac{e}{c}A_x)\psi^*_{n,k_x} = [-\hbar k_x + \frac{eB}{c}y]\,\psi^*_{n,k_x}$$

$$\Rightarrow \quad j_{x\,(n,k_x)} = \frac{1}{m}|\psi_{n,k_x}|^2[\hbar k_x - \frac{e}{c}By] = \frac{\hbar}{m}|\psi_{n,k_x}|^2[y_0(k) - y]$$

$$\Rightarrow \quad I_{x\,(n,k_x)} = \int dy \cdot j_{x(n,k_x)} = \frac{\hbar}{m}\int dy\, |\psi_{n,k_x}|^2[y_0(k) - y]$$

$$= \frac{\hbar}{m}\int dy\,|\phi_n(y)|^2 \underset{\text{even function}}{(\text{-})y}\;\underset{\text{odd}}{\longleftarrow} = 0$$

$y_0(k)$

The ~~total~~ current carried by each state ψ_{n,k_x} is zero.

If imposed with a boundary, how about edge states?

let's consider the upper edge at $y = L_y/2$, in the limit $k_x \to +\infty$

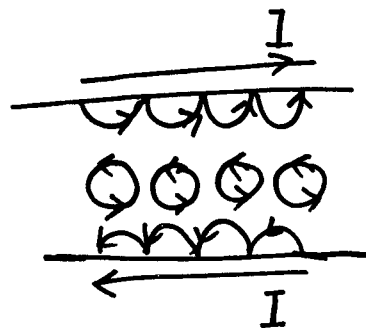$$\psi_{n,k_x} = f_n(y) \frac{1}{\sqrt{L_x}} e^{ik_x x} \,, \quad \text{where } f_n(y) \to \delta(y - L_y/2)$$

$$\Rightarrow (P_x - \frac{e}{c} A_x) \psi_{n,k_x} \simeq (\hbar k_x - \frac{eB}{c}\frac{L_y}{2}) \psi_{n,k_x} \longrightarrow \hbar k_x \psi_{n,k_x}$$

$$\dot{j}_{x, nk_x} \simeq \delta(y - L_y/2) \frac{\hbar k_x}{m} \frac{1}{\sqrt{L_x}} :$$

Edge does carry current!
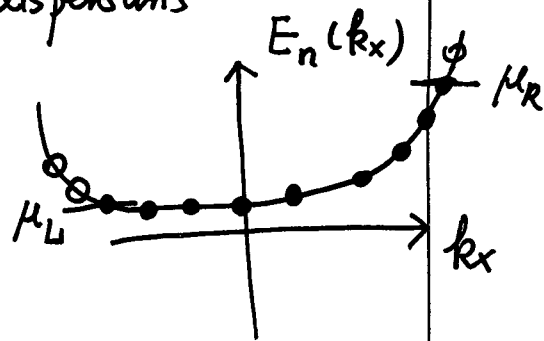
classical picture

$\mathbf{*}$ Now we calculate the Hall conductance

Now considering real sample, such that due to the effect of boundary and impurity, we cannot assume the Landau level energy is exactly flat, but with small dispersions

the group velocity

$$U_{k_x} = \frac{1}{\hbar} \frac{\partial E_n(k_x)}{\partial k_x}$$

$$j_{x, nk_x} = e |\psi_{n,k_x}|^2 U_x \Rightarrow I_{x, nk_x} = \int dy \, e |\psi_{n,k_x}|^2 U_x$$

$$= \frac{e}{L_x} \frac{1}{\hbar} \frac{\partial E_n(k_x)}{\partial k_x}$$

$$\left(\text{assume } \psi_{n,k_x} = f_n(y) \frac{e^{ik_x x}}{\sqrt{L_x}} \quad \text{and} \quad \int dy \, |f_n(y)|^2 = 1 \right)$$

$$\Rightarrow \quad I_{x,n} = \sum_{k_x} I_{x,n,k_x} = \frac{e}{L_x h} L_x \int \frac{dk_x}{2\pi} \frac{\partial E(k_x)}{\partial k_x}$$
$$\text{occupied} \qquad\qquad \text{occupied}$$

$$= \frac{e}{2\pi h} \left[E(\text{right occupied}) - E(\text{left, occupied}) \right] = \frac{e}{h} \cdot \Theta(\mu_R - \mu_L)$$

$$= \frac{e^2}{h} \Delta V_y$$

$\Rightarrow$ each Landau level contribute $\sigma_{xy,n} = \dfrac{e^2}{h}$

$\Rightarrow$ total Hall conductance

$$\boxed{\sigma_{xy} = \sum_{n,\text{occupied}} \sigma_{xy,n} = \frac{\nu e^2}{h}}$$

ν: filling number

Quantum Hall effect

$\nu = 2$