Lect 13 Theory of angular momentum (I)

§1 eigenvalues, eigenstates of angular momentum

$[J_\alpha, J_\beta] = i\,\varepsilon_{\alpha\beta\gamma} J_\gamma$, $D(g) = e^{-i\vec{n}\cdot\vec{J}\theta}$, for $g(\vec{n},\theta)$

we use $|j,m\rangle$ to represent the common eigenstates of

$J^2 = J_x^2 + J_y^2 + J_z^2$, and J_z, such that

$J^2|j,m\rangle = \lambda_j |jm\rangle$ and $J_z|jm\rangle = \lambda_m |jm\rangle$ below.

we will determine the above eigenvalues of $\lambda_j = j(j+1)$ and $\lambda_m = m$

Set $J_\pm = J_x \pm iJ_y$, we have $J_\pm^\dagger = J_\mp$,

its easy to prove that $[J^2, J_\pm] = [J^2, J_z] = 0$.

Ex: check $[J_+, J_-] = 2J_z$ and $J^2 = J_+J_- + J_z(J_z-1)$
$= J_-J_+ + J_z(J_z+1)$

$$J^2 J_\pm |jm\rangle = J_\pm J^2 |jm\rangle = \lambda_j J_\pm |jm\rangle$$

$$[J_z, J_\pm] = [J_z, J_x \pm iJ_y] = iJ_y \pm i(-)iJ_x = \pm(J_x \pm iJ_y) = \pm J_\pm$$

$$J_z J_\pm |jm\rangle = (J_\pm J_z \pm J_\pm)|jm\rangle = (m\pm1) J_\pm |jm\rangle$$

Thus we can start from $|jm\rangle$. and reach

$J_+|jm\rangle, (J_+)^2|jm\rangle, \dots (J_+)^k|jm\rangle$ whose eigenvalues

of J_z are $m+1, \cdots m+k$.

also $J_-|jm\rangle,\ (J_-)^2|jm\rangle,\ \cdots\ (J_-)^{k'}|jm\rangle$, who eigenvalues of J_z are

$m-1,\ m-2,\ \cdots\ m-k'$.

We will show that these two sequences will terminate at finite lengthes.
This is because all the $(J_+)|jm\rangle, \cdots (J_+)^k|jm\rangle,\ J_-|jm\rangle, \cdots (J_-)^{k'}|jm\rangle$ share the same eigen vaue of J^2, ie. λ_j. $J^2 = J_x^2 + J_y^2 + J_z^2 \Rightarrow \lambda_j \geq J_z^2$,
thus $(m+k)^2, (m-k')^2 \leq \lambda_j$, $\Rightarrow$ k and k' must terminate at finite values.

Let us just assume such a sequence with both ends

$$(J_-)^{\underline{k}}|jm\rangle, \cdots \quad J_-|jm\rangle,\ |jm\rangle,\ J_+|jm\rangle, \quad \cdots \quad (J_+)^{\bar{k}}|jm\rangle$$

$\longleftarrow$ $\underline{k}$ terms $\qquad\qquad$ $\longrightarrow$ $\bar{k}$ terms

$(J_+)^{\bar{k}+1}|jm\rangle = 0$ we cannot further apply J_+ on $(J_+)^{\bar{k}}|jm\rangle$,

$(J_-)^{\underline{k}+1}|jm\rangle = 0$ and cannot apply J_- on $(J_-)^{\underline{k}}|jm\rangle$.

from $J^2 = J_-J_+ + J_z(J_z+1) = J_+J_- + J_z(J_z+1)$, we have

$$J^2 (J_+)^{\bar{k}}|jm\rangle = [J_-J_+ + J_z(J_z+1)]\,\widehat{(J_+)^{\bar{k}}}\,|jm\rangle = (m+\bar{k})(m+\bar{k}+1)\{(J_+)^{\bar{k}}|jm\rangle\}$$

$$J^2 (J_-)^{\underline{k}}|jm\rangle = [J_+J_- + J_z(J_z-1)](J_-)^{\underline{k}}|jm\rangle = (m-\underline{k})(m-\underline{k}-1)\{(J_-)^{\underline{k}}|jm\rangle\}$$

$$\Rightarrow \lambda_j^2 = (m+\bar{k})(m+\bar{k}+1) = (m-\underline{k})(m-\underline{k}-1)$$

Because $\bar{k}$ and $\underline{k}$ are positive integers, we have

$$\left.\begin{aligned} m+\bar{k} &= -(m-\underline{k}) \\ m+\bar{k}+1 &= -(m-\underline{k}-1) \end{aligned}\right\} \Rightarrow \quad 2m = \underline{k}-\bar{k},$$

thus m can only take integer, or, half integer values.

Let $j = m+\bar{k} = -(m-\underline{k}) \quad \Rightarrow \quad J^2 = j(j+1)$.

Conculsion: For states $|jm\rangle$ satisfying

$$J^2|jm\rangle = j(j+1)\,|jm\rangle \quad \text{and} \quad J_z|jm\rangle = m|jm\rangle,$$

we have $-j \le m \le j$, and m, j can only be integer, or, half an integer.

$m = -j, -j+1, \cdots j$, takes $2j+1$ possible eigenvalues.

§ normalization and convention of relative phase of $|jm\rangle$.

Consider $|njm\rangle$ which represent a set of orthonormal complete bases for a system. n is another good quantum number, which represents another mechanical observable commutable with J^2, J_z.

$$J_\pm\,|njm\rangle = C_\pm\,|nj\,m\pm1\rangle$$

$$\Rightarrow |C_\pm|^2 = \langle njm|\,J_\mp J_\pm\,|njm\rangle = \langle njm|\,J^2 - J_z(J_z\pm1)\,|njm\rangle$$

$$= j(j+1) - m(m\pm1) = (j\mp m)(j\pm m+1)$$

we fix the phase convention that $C_\pm$ are real $\Rightarrow$

$$\boxed{J_\pm\,|njm\rangle = \sqrt{(j\mp m)(j\pm m+1)}\,\,|nj\,m\pm1\rangle}$$

or

$$\langle n j m{+}1 | J_\pm | n j m \rangle = \sqrt{(j \mp m)(j \pm m + 1)}$$

$$|njm\rangle = (J_-)^{j-m} |njj\rangle \, \frac{\sqrt{(j+m)!}}{\sqrt{(2j)!}\,\sqrt{(j-m)!}}$$

$$= (J_+)^{j+m} |nj{-}j\rangle \sqrt{\frac{(j-m)!}{(2j)!\,(j+m)!}}$$

Important result :

① Assume that an operater K is rotationally invariant, i.e, $[K, \vec{J}] = 0$

then it matrix element $\langle n' jm | K | njm \rangle = f(n, j)$

is independent with m.

Proof. First of all, K is diagonal with respect to j, m.

$$J^2 K |njm\rangle = K J^2 |njm\rangle = j(j+1)\, K |njm\rangle$$

$$J_z K |njm\rangle = m K |njm\rangle$$

$\Rightarrow$ $K|njm\rangle$ shares the same eigenvalues of $j(j+1)$ and m as $|njm\rangle$ does.

$\Rightarrow$ only $\langle n' jm | K | njm \rangle$ can be nonzero, i.e, it can only be non-diagonal with respect to n.

Second, $$\langle n' j m{+}1 | K | n j m{+}1 \rangle = \frac{\langle n' jm | J_- K J_+ | njm \rangle}{(j-m)(j+m+1)}$$

$$= \frac{\langle n' jm | K J_- J_+ | njm \rangle}{(j-m)(j+m+1)}$$

$$J_-J_+ = J^2 - J_z(J_z+1) \Rightarrow J_-J_+|njm\rangle = [j(j+1) - m(m+1)]|njm\rangle$$

$$= (j-m)(j+m+1)\,|njm\rangle$$

$$\Rightarrow \langle n' j\, m+1|\, K\, |n j\, m+1\rangle = \langle n' j m|\, K\, |n j m\rangle \qquad m = -j, \cdots j$$

$\Rightarrow$ $\langle n' j m | K | n j m\rangle$ is independent of m.

(2) Similarly, we can prove if there are two sets of angular momentum eigenstates $|\psi_{jm}\rangle$ and $|\Phi_{jm}\rangle$, we have $\langle \psi_{jm} | \Phi_{jm}\rangle$ is independent of m.

Later, we will see "j" is the quantum number to mark the representation of SU(2) group, and "$m = -j \cdots j$" is the label of the bases in such a representation. The above result is a special case of Wigner-Eackart theorem, what states the above matrix elements are diagonal-blocked with respect to j, and proportional to identity matrix within each diagonal block.