Lect 10: Landau Fermi liquid - microscopic theory

As we learned before, the single fermion Green's function

$$G(x_1, x_2) = -i\langle G_0 | T(\psi(x_1)\psi^\dagger(x_2)) | G_0\rangle, \quad x = (\vec{x}, t).$$

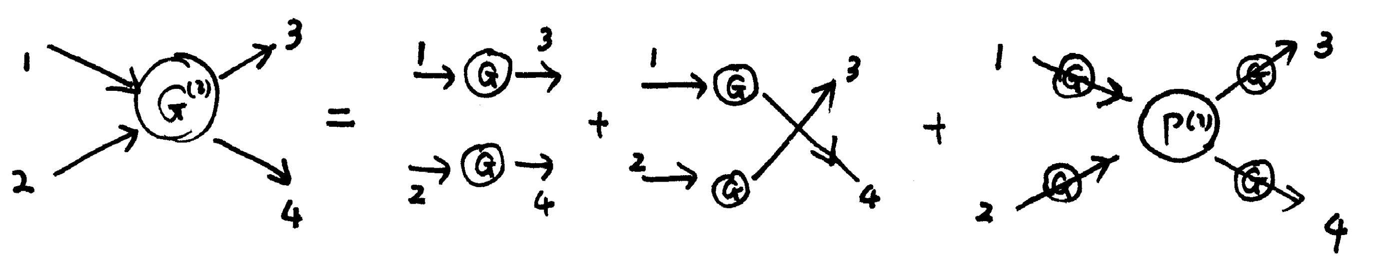
$$\rightarrow \quad G(\vec{p}, \omega) = \frac{1}{\omega - \left[\frac{p^2}{2m} - \mu\right] - \Sigma(\vec{p}, \omega)} = \frac{\left[1 - \frac{\partial\Sigma}{\partial\varepsilon}\right]^{-1}}{\omega - v_F(p - p_F) + i\,\mathrm{sgn}\,\omega} + \cdots$$

the residue at the quasi-particle pole $Z = \left[1 - \frac{\partial\Sigma}{\partial\varepsilon}\right]^{-1}$, and $v_F = \frac{k_F}{m^*}$.

Let us introduce two-body Green's functions

$$G^{(2)}(x_1 x_2; x_3 x_4) = \langle G_0 | T(\psi(x_1)\psi(x_2)\psi^\dagger(x_3)\psi^\dagger(x_4)) | G_0\rangle$$

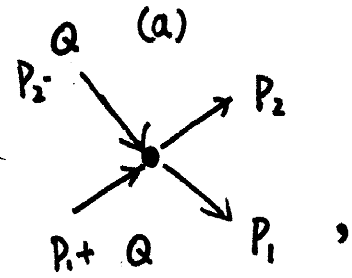
$$= G(1,3)G(2,4) - G(1,4)G(2,3) + i\int d1'd2'd3'd4'\, G(11')G(22')\, \Gamma^{(2)}(1'2'|3'4')\, G(3',3)G(4',4)$$

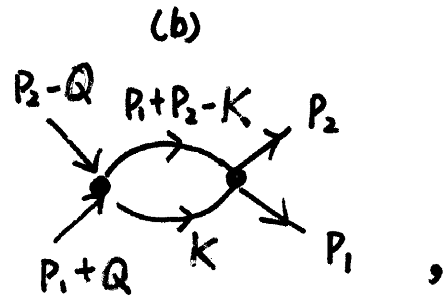


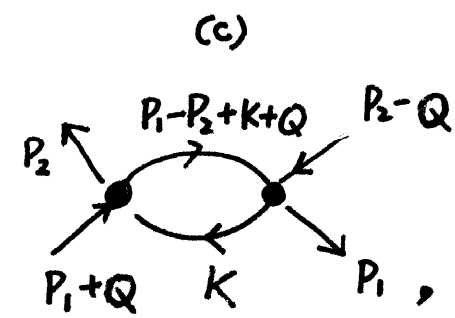
$$\rightarrow \quad G^{(2)}(P_1P_2; P_3P_4) = (2\pi)^8 G(P_1)G(P_2)\left[\delta(P_1 - P_3)\delta(P_2 - P_4) - \delta(P_1 - P_4)\delta(P_2 - P_3)\right]$$

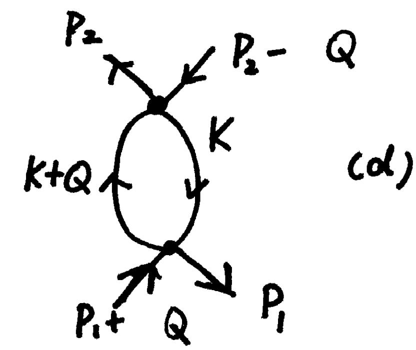
$$+ i\, G(P_1)G(P_2)G(P_3)G(P_4)\, \Gamma^{(2)}(P_1P_2; P_3P_4)$$

define $\Gamma^{(2)}(P_1P_2; P_3P_4) = (2\pi)^4\delta(P_1 + P_2 - P_3 - P_4)\, \Gamma(P_1P_2; P_3P_4)$ ← energy momentum conservation!









First and second order graphs to Γ ← vertex function.

Consisid the vertex function for small values of $(\omega_3-\omega_1, \vec{P}_3-\vec{P}_1)$, or Q

Define $P_3 = P_1 + Q$, $P_4 = P_2 - Q$, and $\Gamma(P_1 P_2 ; Q) \equiv \Gamma(P_1 P_2 ; P_3 P_4)$.

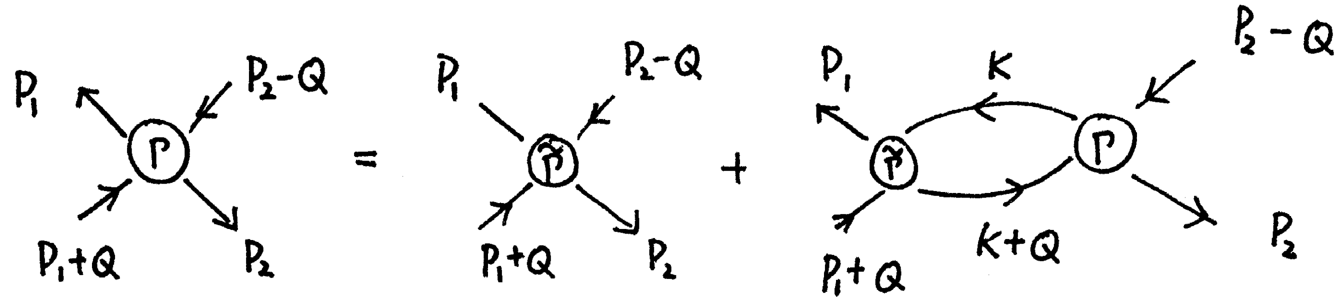
where $Q = (q, \omega)$ is a small 4-vector. This corresponds to nearly forward scattering!

(a) → antisymmetrized matrix elements $v(\vec{Q}) - v(P_1+P_2+\vec{Q})$

(b), (c) have no singularity as $Q \to 0$, because the poles of the two fermion lines do not coalesce. We can safely set $Q = 0$. (internal lines describe large angle scatterings).

However, the diagram d, as $Q \to 0$, the poles of $G(K)G(K+Q)$ are close, we need ~~treat~~ it with care.

We denote $\tilde{\Gamma}$ as the part of Γ, which is no singular as $Q \to 0$, and use $\tilde{\Gamma}(P_1 P_2) = \tilde{\Gamma}(P_1 P_2; Q=0)$.



$$\Gamma(P_1 P_2; Q) = \tilde{\Gamma}(P_1 P_2) - i \int \frac{d^4 k}{(2\pi)^4} \tilde{\Gamma}(P_1; K)\, G(K)\, G(K+Q)\, \Gamma(K P_2; Q)$$

Comes from $(-i)$ $(i)^2$ $(-)$

interaction, each fermi line, fermion loop.

$$K = (k, \varepsilon_k)$$

$$G(K)\,G(Q+K) \approx \frac{z}{\varepsilon_k - v_F(k - k_F) + i\,\mathrm{sgn}\,\varepsilon_k} \; \frac{z}{\omega + \varepsilon_k - v_F(|k+q| - k_F) + i\,\mathrm{sgn}(\omega + \varepsilon_k)}$$

As $q, \omega \to 0$, the singularity approaches $\varepsilon_k = 0$ and $k = k_F$.

we approximate

$$G(K)\,G(Q+K) \xrightarrow{q \to 0} A(\theta)\, \delta(\varepsilon_k)\, \delta(k - k_F) + \phi(K)$$ ← background

$A(\theta)$: the angle between $\vec{k}$ & $\vec{q}$

$$A(\theta) = \int d\varepsilon_k \int dk \; \frac{z}{\varepsilon_k - v_F(k - k_F) + i\,\mathrm{sgn}\,\varepsilon_k} \; \frac{z}{\omega + \varepsilon_k - v_F(|k+q| - k_F) + i\,\mathrm{sgn}(\omega + \varepsilon_k)}$$

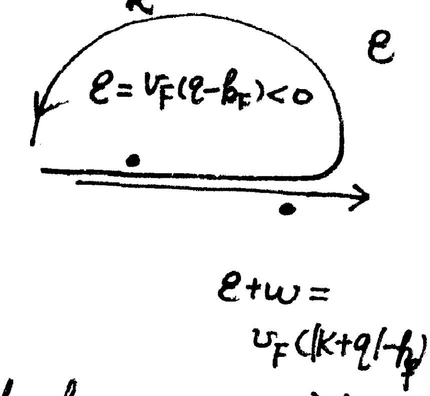
$\boxed{\text{no angular integral}}$

if ε_k & $\varepsilon_k + \omega$ have the same sign $\rightarrow$ $\int d\epsilon = 0$

$|\vec{k}+\vec{q}| \approx k + q\cos\theta$. 1. for $\cos\theta > 0 \Rightarrow k - k_F < 0 < k - k_F + q\cos\theta$

i.e. $k_F - q\cos\theta < k < k_F$

$$A(\theta) \underset{q\to 0}{\sim} \int_{k_F - q\cos\theta}^{k_F} dk \, \frac{2\pi i z^2}{\omega - v_F(|\vec{k}+\vec{q}|-k)}$$

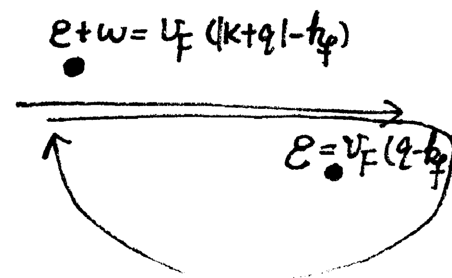


$$= \frac{2\pi i z^2 q\cos\theta}{\omega - v_F q\cos\theta}$$

2. for $\cos\theta < 0 \Rightarrow \quad k + \cos\theta \cdot q - k_F < 0 < k - k_F$

$$\Rightarrow \quad k_F < k < k_F - q\cos\theta$$

$$A(\theta) \underset{q\to 0}{\sim} \int_{k_F}^{k_F - q\cos\theta} dk \, \frac{-2\pi i z^2}{\omega - v_F(|\vec{k}+\vec{q}|-k)}$$



$$= \frac{2\pi i z^2 q\cos\theta}{\omega - v_F q\cos\theta}$$

$$\Rightarrow \quad G(K)\,G(Q+K) = \frac{2\pi i z^2 \vec{q}\cdot\hat{k}}{\omega - v_F \vec{q}\cdot\hat{k}} \, \delta(\varepsilon_k - \mu)\,\delta(k-k_F) + \phi(K)$$

$$\Rightarrow \quad \Gamma(\vec{P}_1, \vec{P}_2; Q) = \tilde{\Gamma}(P_1 P_2) - i\int \frac{d^4k}{(2\pi)^4}\, \tilde{\Gamma}(P_1 K)\,\phi(K)\,\Gamma(K P_2; Q)$$

$$+ \frac{z^2 k_F^2}{(2\pi)^3} \int d\Omega_K \, \tilde{\Gamma}(P_1 K) \frac{\vec{q}\cdot\hat{k}}{\omega - v_F \hat{k}\cdot\vec{q}} \, \Gamma(K P_2; Q)$$

The integral Kenel $\dfrac{\hat{k}\cdot\vec{q}}{\omega-\hat{k}\cdot\vec{q}}$ has non-analytical behavier as $\begin{matrix} q\to 0 \\ \omega\to 0\end{matrix}$

define the limit

$$\Gamma^{\omega}(P_1 P_2) = \lim_{\omega\to 0}\lim_{q\to 0} \Gamma(P_1 P_2 ; Q),$$ under this limit

the kernal $\to 0$. $\Rightarrow$

$$\boxed{\Gamma^{\omega}(P_1 P_2) = \tilde{\Gamma}(P_1 P_2) - i\int \frac{d^4 k}{(2\pi)^4} \tilde{\Gamma}(P_1 K)\,\phi(K)\,\Gamma^{\omega}(K, P_2)}$$

we want to represent $\Gamma(P_1 P_2 ; Q)$ in terms of $\Gamma^{(\omega)}(P_1 P_2)$ which

is also non-singular as $Q\to 0$.

Let us write the above Eq in a compact form

$$\Gamma^{\omega} = \tilde{\Gamma} - i\tilde{\Gamma}\phi\Gamma^{\omega} = \tilde{\Gamma} - i\Gamma^{\omega}\phi\tilde{\Gamma} = (1-i\Gamma^{\omega}\phi)\tilde{\Gamma}$$

need a little excercise

Γ satisfies

$$\Gamma = \tilde{\Gamma} - i\tilde{\Gamma}\phi\Gamma + \tilde{\Gamma}\frac{\vec{q}\cdot\hat{K}}{\omega - v_F\hat{k}\cdot\vec{q}}\Gamma$$

$$\Rightarrow (1-i\Gamma^{\omega}\phi)\Gamma = (1-i\Gamma^{\omega}\phi)\tilde{\Gamma} - i(1-i\Gamma^{\omega}\phi)\tilde{\Gamma}\phi\Gamma + (1-i\Gamma^{\omega}\phi)\tilde{\Gamma}\cdot\frac{\vec{q}\cdot\hat{k}}{\omega - v_F\hat{k}\cdot\vec{q}}\Gamma$$

$$= \Gamma^{\omega} - i\Gamma^{\omega}\phi\Gamma + \Gamma^{\omega}\frac{\vec{q}\cdot\hat{k}}{\omega - v_F\hat{k}\cdot\vec{q}}\Gamma$$

$$\Rightarrow \Gamma = \Gamma^{\omega} + \Gamma^{\omega}\frac{\vec{q}\cdot\hat{k}}{\omega - v_F\hat{k}\cdot\vec{q}}\Gamma,$$ i.e

$$\boxed{\Gamma(P_1 P_2 ; Q) = \Gamma^{\omega}(P_1 P_2) + \frac{z^2 k_F^2}{(2\pi)^3}\int d\Omega\, \Gamma^{\omega}(P_1 K)\frac{\vec{q}\cdot\hat{k}}{\omega - v_F\vec{q}\cdot\hat{k}}\Gamma(K P_2 ; Q)}$$

we are also interested in the other limit $\quad \lim_{k\to0}\lim_{\omega\to0} \frac{\hat{k}\cdot q}{\omega - v_F \hat{k} q} = -\frac{1}{v_F}$.

$$\Gamma^{(k)}(P_1 P_2) = \lim_{k\to0}\lim_{\omega\to0} \Gamma(P_1 P_2; K) \Rightarrow$$

$$\boxed{\Gamma^{(k)}(P_1 P_2) = \Gamma^{\omega}(P_1 P_2) - \frac{z^2}{(2\pi)^3}\frac{k_f^2}{v_F}\int d\Omega\, \Gamma^{(\omega)}(P_1\ K)\,\Gamma^{(k)}(K, P_2)}$$

we are interested on the singular behavior (poles) of $\Gamma(P_1 P_2; Q)$, thus

we neglect $\Gamma^{\omega}(P_1 P_2) \Rightarrow$

$$\Gamma(P_1 P_2; Q) \underset{k\to0}{\simeq} \frac{z^2 k_f^2}{(2\pi)^3}\int d\Omega_K\, \Gamma^{(\omega)}(P_1, K)\frac{\hat{k}\cdot\vec{q}}{\omega - v_F \hat{k}\cdot\vec{q}}\,\Gamma(K P_2; Q)$$

let us fix P_2, and let P_1, P_2 at Fermi surface.

$$\Gamma(\Omega_{P_1}, \Omega_{P_2}; Q) \underset{k\to0}{\simeq} \frac{z^2 k_f^2}{(2\pi)^2}\int d\Omega_K\, \Gamma^{(\omega)}(\Omega_{P_1}, \Omega_K)\frac{\hat{k}\cdot\vec{q}}{\omega - v_F \hat{k}\cdot\vec{q}}\,\Gamma(\Omega_K, \Omega_{P_2}; Q)$$

define

$$u(\hat{p}) = \frac{\hat{p}\cdot\vec{q}}{\omega - \hat{p}\cdot\vec{q}\, v_F}\,\Gamma(\Omega_p, \Omega_{P_2}; Q)$$

$$\Rightarrow (\omega - v_F\hat{p}\cdot\hat{q})\, u(\hat{p}) \simeq (\hat{p}\cdot\vec{q})\frac{z^2 k_f^2}{(2\pi)^3}\int d\Omega_K\, \Gamma^{(\omega)}(\Omega_{p}, \Omega_K)\, u(\hat{k})$$

$$\left[\frac{\omega}{v_F q} - \cos\theta\right] u(\hat{p}) = \cos\theta\, \frac{m^* k_f}{\pi^2}\frac{1}{2}\int \frac{d\Omega_K}{4\pi}\, \boxed{z^2\, \Gamma^{(\omega)}(\Omega_{p}, \Omega_K)}\, u(\hat{k})$$

Compared with zero sound eigen-equation $\Longrightarrow$ Landau interaction parameter

$$\boxed{f(P_1, P_2) = z^2\, \Gamma^{(\omega)}(\hat{P}_1\ \hat{P}_2)}$$

Γ^ω represents virtual excitation with small energy transfer. Γ^k describes the physical forward scattering amplitude $(p_1 p_2 \to p_1 p_2)$

From

$$\Gamma^k(p_1,p_2) = \Gamma^\omega(p_1,p_2) - \frac{z^2}{(2\pi)^3}\frac{k_F^2}{v_F}\int d\Omega_K\, \Gamma^\omega(p_1,K)\,\Gamma^{(k)}(K,p_2)$$

$$\Rightarrow z^2\Gamma^k(p_1,p_2) = f(p_1,p_2) - \frac{k_F m^*}{\pi^2}\frac{1}{2}\int\frac{d\Omega}{4\pi}\, f(p_1,K)\, z^2\Gamma^k(K,p_2)$$

restore spin

$$z^2\Gamma^k_{\sigma\sigma'}(p_1,p_2) = f(p_1\sigma_1;p_2\sigma_2) - N_0\int\frac{d\Omega}{4\pi}\, f(p_1\sigma_1;K\sigma_3)\, z^2\Gamma^k(K\sigma_3;p_2\sigma_2)$$

$\Rightarrow$ harmonics decomposition

$$N_0\,(z^2\Gamma^k_{\sigma\sigma'}(p_1,p_2)) = \sum_l \left[B_l + C_l\,\sigma\cdot\sigma'\right] P_l(\cos\theta) \qquad \cos\theta = \hat{p}_1\cdot\hat{p}_2$$

$$\Rightarrow \quad B_l = \frac{F_l^s}{1+\frac{F_l^s}{2l+1}}, \qquad C_l = \frac{F_l^a}{1+\frac{F_l^a}{2l+1}}$$

Sum rule

$$\boxed{\lim_{p'\to p}\Gamma^k(p'\sigma,p\sigma) = 0}$$

same spin

$$\Rightarrow \boxed{0 = \sum_L (B_L + C_L) = \sum_L\left(\frac{F_l^s}{1+\frac{F_l^s}{2L+1}} + \frac{F_l^a}{1+\frac{F_l^a}{2L+1}}\right)}$$