Lect 7   Boltzmann   transport (I)

§ Wigner distribution

we need to consider a spatially inhomogenous (slowly varing) system. We will talk about the particle with momentum $\vec{p}$ at point $\vec{r}$, and the distribution function $n_{\alpha\beta}(p; r, t)$ in the phase space. A more rigourus definition is as follows.

define $f(p; k; t) = \langle C^{\dagger}_{p+k/2} \, C_{p-k/2} \rangle(t)$, and perform the Fourier transform over small variable k, and arrive at

$$n_{\alpha\beta}(p; r, t) = \sum_{k} f(p; k; t) \, e^{ikr}$$

another way to express $n_{\alpha\beta}(p; r, t) = \int dr' \, e^{ip \cdot r'} \langle \psi^{\dagger}_{\alpha}(r + \frac{r'}{2}) \, \psi_{\beta}(r - \frac{r'}{2}) \rangle$

where "r" is the center of mass coordinate, r' is the relative coordinate.

$n_{\alpha\beta}(p; r, t)$ is a semiclassical distribution. The resolution of Δp and Δr need to satisfy $\Delta p \cdot \Delta r \geq \hbar/2$.

§ Boltzmann equation:

Let us study the equation of motion of $n(p; r; t)$. It has three contributions: ① Flow in the momentum space

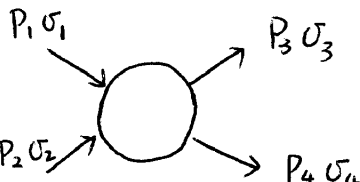② Flow in the real space

③ Collisions

$$\frac{\partial n(p;r;t)}{\partial t} + \nabla_r \left[v_p(r,t)\, n(p;r;t) \right] + \nabla_p \left[f_p(r,t)\, n(p;r;t) \right] = I_{collision}$$

$$\uparrow dr/dt \qquad\qquad \uparrow dp/dt$$

where $v_p(r,t) = \nabla_p \mathcal{E}_p(r,t)$, $f_p(r,t) = -\nabla_r \mathcal{E}_p(r,t)$

$$\Rightarrow \quad \boxed{\frac{\partial}{\partial t} n(p;r;t) + \nabla_p \mathcal{E}(p,rt)\, \nabla_r n(p\,rt) - \nabla_r \mathcal{E}(prt)\, \nabla_p n(prt) = I_{collision}}$$

Collision integral

$$\frac{2\pi}{\hbar} |\langle 34|t|12\rangle|^2 \, \delta(\mathcal{E}_1 + \mathcal{E}_2 - \mathcal{E}_3 - \mathcal{E}_4) \, n_1 n_2 (1-n_3)(1-n_4)$$

$$\frac{2\pi}{\hbar} |\langle 12|t|34\rangle|^2 \, \delta(\mathcal{E}_1 + \mathcal{E}_2 - \mathcal{E}_3 - \mathcal{E}_4) \, n_3 n_4 (1-n_1)(1-n_2)$$

define $\quad \frac{2\pi}{\hbar} |\langle 34|t|12\rangle|^2 = \frac{1}{V^2} W(12;34) \, \delta_{p_1+p_2,\, p_3+p_4} \, \delta_{\sigma_1+\sigma_2 = \sigma_3+\sigma_4}$

$$\Rightarrow \mathcal{I}_{coll}[n_p] = \frac{1}{V^2} \sum_{p_2 \sigma_2} \sum_{p_3 \sigma_3} \sum_{p_4 \sigma_4} W(12;34) \, \delta_{p_1+p_2,\, p_3+p_4} \, \delta_{\sigma_1+\sigma_2,\, \sigma_3+\sigma_4}$$

$$\delta(\mathcal{E}_1 + \mathcal{E}_2 - \mathcal{E}_3 - \mathcal{E}_4) \left[n_3 n_4 (1-n_1)(1-n_2) - n_1 n_2 (1-n_3)(1-n_4) \right]$$

In most situations, we will use a relaxation time approximation

$$\mathcal{Q}\left[\mathcal{I}(n_p)\right]_{RT} = -\frac{\delta n}{\tau}$$

In the case of $\omega \tau \gg 1$, we can neglect the collision integral.

boxed: Warm up. — the Boltzman Eq for the density channel, for ordinary

● FL. — only density mode

$$\frac{\partial}{\partial t} n(r p t) + \nabla_p \varepsilon (r p t) \nabla_r n(r p t) - \nabla_r \varepsilon(r p t) \nabla_p n(r p t) = I_{coll}$$

linearing the equation

$$\varepsilon(r p t) = \varepsilon_0(\varphi) + \frac{1}{V} \sum_{p'} f^S(\varphi, p') \delta n_{p'}(r, t)$$

$$n(r p t) = n_0(\varphi) + \delta n(r, p, t)$$

$\nabla_r n(r, p, t)$ and $\nabla_r \varepsilon(r, p, t)$ are already at the order of δn

thus we keep $\nabla_p \varepsilon(r, p t) = \nabla_p \varepsilon_0(\varphi)$ and $\nabla_p n(r p t) = \nabla_p n_0(\varphi)$

at the zeroth order.

●

$$\Rightarrow \frac{\partial}{\partial t} \delta n(r, p, t) + \vec{v}_p \cdot \nabla_r \delta n(r, p, t) - \nabla_p n_0(\varphi) \int \frac{d\vec{p}'}{(2\pi)^3} f^S_{\vec{p}\vec{p}'} \nabla_r \delta n^{p'}(r, t) = 0$$

$$\frac{\partial}{\partial t} \delta n(r, p, t) + \vec{v}_p \cdot \nabla_r \left[\delta n(r p, t) - \frac{\partial n_0(\varphi)}{\partial \varepsilon} \int \frac{d\vec{p}'}{(2\pi)^3} f^S_{\vec{p}\vec{p}'} \delta n(r, p', t) \right] = 0$$

do Fourier transform

$$\delta n(r p t) = \sum_q \delta n(\vec{p}) e^{i(qr - \omega t)}$$

← boxed: Collision neglected at $\omega \tau \gg 1$

$$\Rightarrow (-i\omega + i \vec{v}_p \cdot \vec{q}) \delta n(\vec{p}) - \frac{\partial n_0(\varphi)}{\partial \varepsilon} \vec{v}_p \cdot i\vec{q} \left(\int \frac{d\vec{p}'}{(2\pi)^3} f^S_{\vec{p}\vec{p}'} \delta n(p') \right) = 0$$

●

$$(\omega - v_f q \cos\theta_p) \delta n(\vec{p}) + \frac{\partial n_0(\varphi)}{\partial \varepsilon} v_f q \cos\theta_p \left(\int \frac{d\vec{p}'}{(2\pi)^3} f^S(\varphi \vec{p}') \delta n(\varphi') \right) = 0$$

$$(S - \cos\Theta_p)\, \delta n(\vec{p}) - \left(-\frac{\partial n_0(p)}{\partial \mathcal{E}_p}\right) \cos\Theta_p \int \frac{d^3p'}{(2\pi)^3}\, f^s(\vec{p},\vec{p}')\, \delta n(\vec{p}) = 0$$

define $\quad \delta n(\hat{p}) = \int \frac{p^2 dp}{(2\pi)^3}\, \delta n(\vec{p}) \quad$ —— integrate over radial

$$\Rightarrow (S - \cos\Theta_p) \int \frac{p^2 dp}{(2\pi)^3}\, \delta n(\vec{p}) - \cos\Theta_p \int \frac{p^2 dp}{(2\pi)^3} \int \frac{p'^2 dp' \, d\Omega_{p'}}{(2\pi)^3}\, f^s(\hat{p},\hat{p}')\, \delta n(\vec{p}') = 0$$

$$\underbrace{\left(-\frac{\partial n_0(p)}{\partial \mathcal{E}_p}\right)}$$

$$(S - \cos\Theta_p)\, \delta n(\hat{p}) - \cos\Theta_p \int \frac{p^2 dp}{(2\pi)^3} \left(-\frac{\partial n_0(p)}{\partial \mathcal{E}_p}\right) \int d\Omega_{p'}\, f^s(\hat{p},\hat{p}')\, \delta n(\hat{p}') = 0$$

$$\underbrace{\int \frac{p^2 dp}{(2\pi)^3}\, \delta(\mathcal{E}_p - \mathcal{E}_f)} = \frac{N_0}{4\pi} \leftarrow \text{single spinor} \qquad DoS$$

$$\Rightarrow \boxed{(S - \cos\Theta_p)\, \delta n(\hat{p}) - N_0 \cos\Theta_p \int \frac{d\Omega_{p'}}{4\pi}\, f^s(\hat{p},\hat{p}')\, \delta n(\hat{p}') = 0}$$

next step is another Fourier transform

$$\delta n(\hat{p}) = \sum_\ell Y_{\ell 0}(\hat{p})\, u_\ell \qquad (\text{set the direction } \hat{q} \text{ as } \hat{z}\text{-axis})$$

$$\sum_\ell (S - \cos\Theta_p) Y_{\ell 0}(\hat{p})\, u_\ell - N_0 \cos\Theta_p \int \frac{d\Omega_{p'}}{4\pi} \sum_{\ell'm'} f^s_{\ell'} \frac{4\pi}{2\ell+1} Y_{\ell'm'}(\hat{p}) Y^*_{\ell'm'}(\hat{p}')$$

$$\sum_\ell Y_{\ell 0}(\hat{p}')\, u_\ell = 0$$

$$\boxed{\sum_\ell Y_{\ell 0}(\hat{p})\, u_\ell - \frac{\cos\Theta_p}{S - \cos\Theta_p} \sum_\ell \frac{F^s_\ell}{2\ell+1} Y_{\ell 0}(\hat{p})\, u_\ell = 0}$$

$$\bullet \int d\Omega_p \, Y^*_{\ell'm'}(\hat{p}) \, \sum_{\ell} \, Y_{\ell 0}(\hat{p}) \, u_\ell \; - \; \sum_{\substack{\ell\ell' \\ m'}} \int d\Omega_p \, \frac{\cos\theta_p}{S-\cos\theta_p} \, Y^*_{\ell'm'}(\hat{p}) \, \frac{F_\ell^S}{2\ell+1} \, Y_{\ell 0}(\hat{p}) \, u_\ell = 0$$

exchange $\ell \leftrightarrow \ell'$

$$\Rightarrow \quad u_\ell \; - \; \sum_{\ell'} \int d\Omega_p \, Y^*_{\ell 0}(\hat{p}) \, \frac{F_{\ell'}^S}{2\ell'+1} \, Y_{\ell' 0}(\hat{p}) \; \frac{\cos\theta_p}{S-\cos\theta_p} \, u_{\ell'} = 0$$

$$\boxed{\begin{aligned} &\frac{u_\ell}{\sqrt{2\ell+1}} + \sum_{\ell'} F_{\ell'}^S \, \Omega_{\ell\ell'}(S) \, \frac{u_{\ell'}}{\sqrt{2\ell'+1}} = 0 \\[2mm] &\text{where} \quad \Omega_{\ell\ell'}(S) = \frac{-1}{\sqrt{(2\ell'+1)(2\ell+1)}} \int d\Omega_p \, Y^*_{\ell 0}(\hat{p}) \, Y_{\ell' 0}(\hat{p}) \, \frac{\cos\theta_p}{S-\cos\theta_p} \end{aligned}}$$

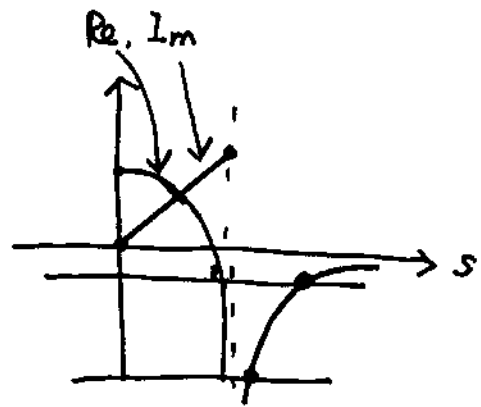① truncate at $\ell=0$ $\Rightarrow$

$$u_0 + F_0^S \, \Omega_{00}(S) \, u_0 = 0 \quad \Rightarrow \quad \frac{-1}{F_0^S} = \Omega_{00}(S)$$

$$\Omega_{00}(S) = \int \frac{d\Omega_p}{4\pi} \frac{-\cos\theta_p}{S-\cos\theta_p} = \int \frac{d\Omega_p}{4\pi} \left[\frac{-S}{S-\cos\theta_p + i\eta} + 1 \right] = + \left[1 + \frac{S}{2} \ln\left| \frac{S-1}{S+1} \right| \right]$$
$$+ i \frac{\pi}{2} S \, \Theta(1-S)$$

$$\Rightarrow \quad \boxed{\frac{1}{-F_0^S} = 1 + \frac{S}{2} \ln\left| \frac{S-1}{S+1} \right|}$$

for $F_0^S \to 0^+$, $S \to 1^+$

$$1 + \frac{1}{2} \ln\frac{S-1}{2} = -\frac{1}{F_0^S} \Rightarrow S \approx 1 + 2 e^{\frac{-2}{F_0^S}}$$

$\bullet$ for $F_0^S \gg 1$ $\Rightarrow$ $1 + \frac{S}{2} \ln\left| \frac{S-1}{S+1} \right| \sim -\frac{1}{3S^2} \Rightarrow S = \sqrt{\frac{F_0}{3}}$

(S is the sound velocity)

§ Spin-related transport

define $\quad n_p(r,t) = \frac{1}{2} tr [n_{p,\alpha\beta}], \quad \vec{\sigma}_p^{(r,t)} \frac{1}{2} tr [n_{p,\alpha\beta} \vec{\tau}_{\beta\alpha}]$

$\Rightarrow \boxed{n(p;rt) = n_p(r,t) \delta_{\alpha\alpha'} + \vec{\sigma}_p(r,t) \cdot \vec{\tau}_{\alpha\alpha'}} \quad (\begin{array}{l} \text{decompose into} \\ \text{density and spin} \end{array})$

the quasi-particle energy can be written as

$$\boxed{\varepsilon (prt) = \varepsilon_p(r,t) \delta_{\alpha\alpha'} + \vec{h}_p(r,t) \cdot \vec{\tau}_{\alpha\alpha'}.}$$

Plug in the Boltzman Eq

$$\frac{\partial n(rpt)}{\partial t} + \frac{\partial}{\partial r} \left[\frac{\partial \varepsilon}{\partial p} \cdot n(rpt) \right] + \frac{\partial}{\partial p} \left[-\frac{\partial \varepsilon}{\partial r} \cdot n(r.pt) \right]$$

Need symmetrizization

$$- \frac{1}{i\hbar} [n(r,pt), \varepsilon(r,p,t)] = I_{collision}$$
$$\underbrace{\qquad\qquad\qquad\qquad}_{\text{Larmor procession}}$$

Seperate variables

$$\boxed{\begin{aligned} &\frac{\partial n_p(rt)}{\partial t} + \frac{\partial}{\partial r_i} \left[\frac{\partial \varepsilon_p}{\partial p_i} n_p + \frac{\partial \vec{h}_p}{\partial p_i} \cdot \vec{\sigma}_p \right] + \frac{\partial}{\partial p_i} \left[-\frac{\partial \varepsilon}{\partial r_i} n_p - \frac{\partial \vec{h}_p}{\partial r_i} \cdot \vec{\sigma}_p \right] = I_{coll} \\[2mm] &\frac{\partial \vec{\sigma}_p(rt)}{\partial t} + \frac{\partial}{\partial r_i} \left[\frac{\partial \varepsilon_p}{\partial p_i} \vec{\sigma}_p + \frac{\partial \vec{h}_p}{\partial p_i} n_p \right] + \frac{\partial}{\partial p_i} \left(-\frac{\partial \varepsilon}{\partial r_i} \vec{\sigma}_p - \frac{\partial \vec{h}_p}{\partial r_i} \cdot n_p \right) = \frac{2}{\hbar} \vec{h}_p \times \vec{\sigma}_p \\ &\hspace{11cm} + I_{coll.} \end{aligned}}$$

Lamor precession: in an external magnet field $\mathcal{H}$, it couples to

electron spin as $-\frac{\gamma}{2}\mathcal{H}\cdot\sigma$ $\Rightarrow$ $\frac{\partial\vec{\sigma}_p}{\partial t} = \vec{\sigma}_p \times \gamma\mathcal{H}$ $\Rightarrow$ $\omega_0 = \gamma\mathcal{H}$

In the Fermi liquid, $h_p = -\gamma\frac{\hbar}{2}\mathcal{H} + 2\int\frac{d^3p'}{(2\pi)^3} \quad f^a(p,p')\,\sigma_{p'}$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \hookrightarrow$ Contribution from interaction

in the uniform system, we have

$$\frac{\partial\vec{\sigma}_p}{\partial t} = \gamma\,\vec{\sigma}_p \times \vec{\mathcal{H}} - \frac{4}{\hbar}\int\frac{d^3p'}{(2\pi)^3}\,(\vec{\sigma}_p \times \vec{\sigma}_{p'})$$

define $\delta\;\;\sigma(\hat{n}) = 2\int\frac{d^3p}{(2\pi)^3}\cdot p^2\sigma(p,\hat{n})$, (we integrate out radius direction).

$$\Rightarrow \quad 2\int\frac{p^2 dp}{(2\pi)^3}\frac{\partial\vec{\sigma}(p,\hat{n})}{\partial t} = 2\int\frac{p^2 dp}{(2\pi)^3}\,\vec{\sigma}(p,\hat{n}) \times (\gamma\vec{\mathcal{H}}) - 8\int\frac{d\Omega'}{4\pi}\int\frac{p^2 dp'}{(2\pi)^3}\int\frac{p^2 dp}{(2\pi)^3}\,\sigma_p \times \sigma_{p'}\quad f^a(p,p')$$

$$\frac{\partial}{\partial t}\sigma(\hat{\Omega}_p) = \gamma\,\sigma(\hat{\Omega}_p) \times \vec{\mathcal{H}} - \frac{2}{\hbar}\int\frac{d\Omega'_p}{4\pi}\,f^a(p,p')\,\hat{\sigma}(\Omega_p)\,\hat{\sigma}(\Omega_{p'})$$

In the external field $\sigma(\hat{\Omega}_p) = \sigma^0 + \delta\sigma(\hat{p})$ $\Rightarrow$

$$\frac{\partial}{\partial t}\delta\vec{\sigma}(\Omega_p) = \gamma\,\delta\vec{\sigma}(\Omega_p) \times \vec{\mathcal{H}} - \frac{2}{\hbar}\int\frac{d\Omega'_p}{4\pi}\,f^a(p,p')\left[\delta\sigma(\hat{p})\times\sigma^0 + \sigma^0\times\delta\sigma(\hat{p}')\right]$$

define $\delta\sigma_+(\hat{\Omega}_p) = \delta\sigma_x(\hat{\Omega}_p) + i\,\delta\sigma_y(\hat{\Omega}_p)$

$$\frac{\partial}{\partial t}\delta\sigma_+(\Omega_p) = -i\left[\omega_0\,\delta\sigma_+(\Omega_p) - \frac{2\sigma^0}{\hbar}\int\frac{d\Omega'}{4\pi}\,f^a(p,p')\,(\delta\sigma_+(\Omega_p) - \delta\sigma_+(\Omega_{p'}))\right]$$

$$= -i\left[(\omega_0 - \frac{2}{\hbar}N(0)^{-1}F_0^a\,\sigma^0)\,\delta\sigma_+(\Omega_p) + \frac{2}{\hbar}\sigma^0\int\frac{d\Omega'}{4\pi}\,f^a(p,p')\,\delta\sigma_+(\Omega_{p'})\right]$$

expand $\delta\sigma_+(\Omega_p) = \sum_{\ell m}\delta\sigma_+(\ell m)\,Y_{\ell m}(\Omega_p)$

$$\int \frac{d\Omega'}{4\pi} f^a(p,p') \, \delta\sigma_+(\Omega_{p'}) = N(0) \int \frac{d\Omega'_p}{4\pi} F_\ell^a \frac{4\pi}{2\ell+1} Y_{\ell m}(\Omega_p) Y_{\ell m}^*(\Omega_{p'}) \sum_{\ell m'} Y_{\ell' m'}^{(\Omega_{p'})} \delta\sigma_+(\ell'm')$$

$$= N(0) \sum_{\substack{\ell \\ m}} \frac{F_\ell^a}{2\ell+1} Y_{\ell m}(\Omega_p) \, \delta\sigma_+(\ell m)$$

$$\Rightarrow \quad \frac{\partial}{\partial t}\delta\sigma_+(\ell m) = -i\left\{\omega_0 - \frac{2}{\hbar} N(0) F_0^a \sigma^\circ \quad + N(0) \frac{F_\ell^a}{2\ell+1}\right\} \delta\sigma_+(\ell m)$$

$$\Rightarrow \quad \omega_{\ell+} = \left\{\omega_0 - \frac{2}{\hbar}\sigma_0 N(0)\left[F_0^a - \frac{F_\ell^a}{2\ell+1}\right]\right\}$$

$$\sigma^0 = \frac{\gamma\hbar}{2}\frac{N(0)}{1+F_0^a}\mathcal{H}, \quad \omega_0 = \gamma\mathcal{H}$$

$$\Rightarrow \quad \boxed{\frac{\omega_{\ell+}}{\omega_0} = \frac{1+F_\ell^a/2\ell+1}{1+F_0^a}}$$

the $\ell=0$ channel Larmor frequency
is not modified by interaction
because Spin is conserved by interaction!

§spin hydrodynamic equations

Integrate over momentum for the Boltzman transport equation $\Rightarrow$

$$\frac{\partial}{\partial t}\vec\sigma(r,t) + \frac{\partial}{\partial r_i}\vec{j}_i(\vec r,t) = -\frac{2}{\hbar}\int\frac{d\vec p}{(2\pi)^3}\vec\sigma_p^{(r,t)}\times\left[-\frac{\gamma}{2}\mathcal{H} + \int\frac{d\vec p'}{(2\pi)^3}f^a(p,p')\sigma_p'\right]$$

$$\boxed{\frac{\partial}{\partial t}\vec\sigma(r,t) + \frac{\partial}{\partial r_i}\vec{j}_i(\vec r,t) = \gamma\vec\sigma(r,t)\times\mathcal{H}(r,t)} \quad \text{(interaction part cancels).}$$

where $\quad \vec\sigma(r,t) = 2\int\frac{d^3p}{(2\pi)^3}\vec\sigma(r,p,t)$

$$\vec{j}_i(r,t) = 2\int\frac{d^3p}{(2\pi)^3}\left[\frac{\partial\varepsilon_p}{\partial p_i}\vec\sigma(r,p,t) + \frac{\partial\vec h_p}{\partial p_i}n_p(r,p,t)\right]$$

$$\vec{j}_i(r,t) = 2\int \frac{d^3p}{(2\pi)^3} \left[\frac{\partial \mathcal{E}_p^0}{\partial p_i} \vec{\sigma}(r,p,t) - \frac{\partial n_p^0}{\partial p}(r,p,t) \vec{h}_p \right] \quad (\text{linearize}).$$

$$= 2\int \frac{d^3p}{(2\pi)^3} \; v_{p_i} \left(\vec{\sigma}_p - \frac{\partial n_p^0}{\partial \mathcal{E}_p} \vec{h}_p \right)$$

v_pi is odd function, after integration, this term goes to zero

$$= 2\int \frac{d^3p}{(2\pi)^3} \; v_{p_i} \left(\vec{\sigma}_p - \frac{\partial n_p^0}{\partial \mathcal{E}_p} \left(-\frac{\gamma}{2}\frac{\hbar}{2}\mathcal{H} + 2\int \frac{d^3p'}{(2\pi)^3} f^a(p,p') \vec{\sigma}_{p'} \right] \right)$$

$$= 2\int \frac{d^3p}{(2\pi)^3} \; v_{p_i} \vec{\sigma}_p \quad - \quad 4\int \frac{d^3p}{(2\pi)^3} \; v_{p_i} \frac{\partial n_p^0}{\partial \mathcal{E}_p} \int \frac{d^3p'}{(2\pi)^3} f^a(p,p') \vec{\sigma}_{p'}$$

$$2\int \frac{d^3p}{(2\pi)^3} \; v_{p_i} \frac{\partial n_p^0}{\partial \mathcal{E}_p} f^a(p,p') = -N(0) \int \frac{d\Omega}{4\pi} \sum_\ell f_\ell^a \quad P_\ell(\cos\theta) \; v_F \cdot \cos\theta \quad \left(\begin{array}{l} \text{set } p' \\ \text{along } z \\ \text{axis} \end{array} \right).$$

$$= -\frac{F_1^a}{3} v_F$$

$$\Rightarrow \quad \vec{j}_i(r,t) = 2\int \frac{d^3p}{(2\pi)^3} \; v_{p_i} \vec{\sigma}_p \; + 2\int \frac{d^3p'}{(2\pi)^3} \; \frac{F_1^a}{3} v_{p_i}' \vec{\sigma}_{p'}$$

$$\boxed{\vec{j}_i(r,t) = 2\int \frac{d^3p}{(2\pi)^3} \; v_{p_i} \vec{\sigma}_p \left(1 + \frac{F_1^a}{3} \right)}$$