Lect 5: Fermi liquid theory (I)

Landau: 1956 ^3He normal state:

hard core radius ~ 2.5 Å, average interparticle distance ~ 3.5 Å

we assume that there are no phase transitions: crystalline order, magnetic order, superfluidity, ...

§1 Concept of quasi-particles

Suppose we have a N-body ground state $|G\rangle$. At time $t=0$, an extra particle is inserted in the plane wave state $C_k^\dagger$. After a time period of T, we check the amplitude of such a particle still in the state of $\vec{k}$ at time T.

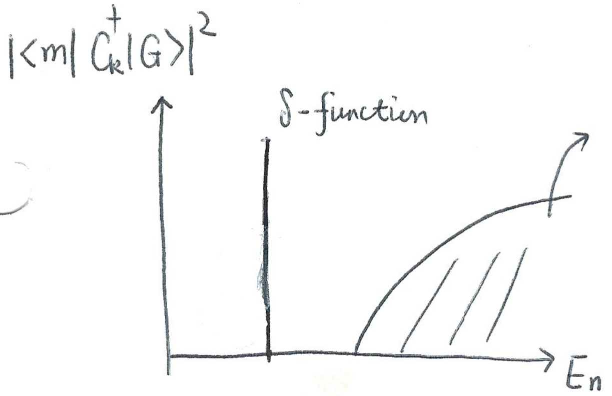
$$e^{-iHT} C_k^\dagger |G\rangle, \quad \text{the inner product with} \quad C_k^\dagger e^{-iHT} |G\rangle$$

$$\rightarrow G_k(T) = \langle G| e^{iHT} C_k \, e^{-iHT} C_k^\dagger | G\rangle = \langle G| C_k(T) \, C_k^\dagger(0) | G\rangle$$

we expand $G_k(T)$ in terms of Lehman representation as

$$G_k(T) = \sum_m \langle G| C_k(T) |m\rangle\langle m| C_k^\dagger(0)| G\rangle = \sum_m |\langle G| C_k |m\rangle|^2 \, e^{-i(E_m - E_g)T}$$

In many cases, the spectra weight of $|\langle m| C_k^\dagger |G\rangle|$ behaves like



$$G_k(T) = Z\, e^{-i E_k' T} + \sum_n |\langle G | C_k | n\rangle|^2 e^{-i E_n T}.$$

The second part represents a continuum in the energy space, this leads to a rapid decaying function.

The long time behavor is controlled by the one particle-like excitation. In the Fermi liquid state $0 < Z < 1$. The δ-function spike can be considered as quarsiparticle state.

§2: Adiabatical continuity:

Each state of the free fermi gas corresponds to a state of the interacting system by turning on the interaction adiabatically.

At zero temperature, the quasiparticle distribution satisfies

$$n^0_{p\sigma} = \begin{cases} 1 & k \le k_F \\ 0 & k > k_F . \end{cases}$$

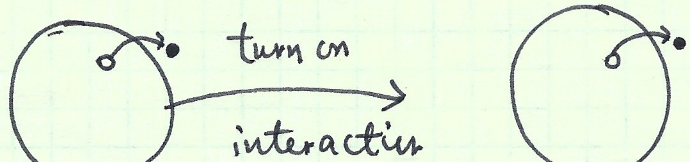
We can creat excitations by moving some quarsi-particles inside the fermi sphere into states outside the fermi surface. The quasi-particle energy is defined as

$$E - E_0 = \sum_{p\sigma} \epsilon(p)\, \delta n_{\vec{p},\vec{\sigma}}, \quad \text{so do}$$

$$\vec{P} = \sum_{p\sigma} \vec{p}\, \delta n_{p,\sigma}, \quad \vec{S} = \sum_{\sigma} \vec{\sigma}\, \delta n_{p\sigma}$$

Please note that $\vec{P}_{tot}$ and $\vec{S}_{tot}$ are conserved quantities through the process of turning interaction.

Fermi gas



Fermi liquid

quasi-particle / quasi-hole

bare electron (particle) / hole

let us expand $\boxed{\mathcal{E}(k) = \left(\frac{d\mathcal{E}}{dk_f}\right)\Big|_{k=k_f} (k-k_f), \quad \text{i.e. } v_F = \left(\frac{d\mathcal{E}}{dk}\right)_{k_f}}$

$\boxed{\text{define effective mass } m^* = P_F / v_F}$

§ Interactions among quasi-particles

We expand the variation of the ground state energy to the 2nd order of $\delta n_{p\sigma}$,

$$\delta E = \sum_{p\sigma} \frac{\delta E}{\delta n_{p\sigma}} \delta n_{p\sigma} + \frac{1}{2} \sum_{p\sigma, p'\sigma'} \frac{\delta^2 E}{\delta n_{p\sigma} \delta n_{p'\sigma'}} \delta n_{p\sigma} \delta n_{p'\sigma'}$$

$$\mathcal{E}_{p,\sigma} = \frac{\delta E}{\delta n_{p\sigma}},$$

$$\frac{1}{V} f(\vec{p}, \vec{p}'; \sigma\sigma') = \frac{\delta^2 E}{\delta n_{p\sigma} \delta n_{p'\sigma'}} \longrightarrow \text{the unit of } f = \text{energy} \times \text{volume}$$

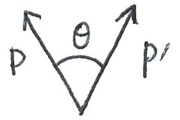
i.e Fourier component of interaction

$$\Rightarrow \boxed{\delta E = \sum_{p\sigma} \mathcal{E}_{p\sigma} \delta n_{p\sigma} + \frac{1}{2V} \sum_{p\sigma, p'\sigma'} f(\vec{p}\vec{p}'; \sigma\sigma') \delta n_{p\sigma} \delta n_{p'\sigma'}}$$

$\boxed{f(\vec{p}\vec{p}'; \sigma\sigma') : \text{Landau interaction function}}$

we have

$$f(pp';\sigma\sigma') = f_s(\cos\theta) + f_a(\cos\theta)\,\sigma\sigma'$$



where $f_s = (f_{\uparrow\uparrow} + f_{\uparrow\downarrow})/2$

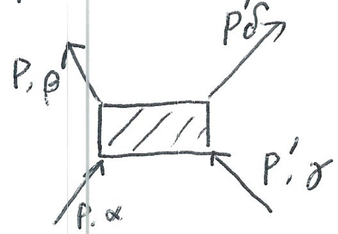
$$f_a = (f_{\uparrow\uparrow} - f_{\uparrow\downarrow})/2$$

More generally, spin does not need to be diagnoal, and should be represented as density matrix $\delta n_{p,\alpha\beta}$, and physical quantities, such as spin, should be represented as $S = tr[\vec{\sigma}\,\delta n_p] = (\vec{\sigma})_{\beta\alpha}\,\delta n_{p,\alpha\beta}$.

the interaction function most generally shoulel be

$$\frac{1}{2V}\sum_{\substack{pp'\\ \alpha\beta,\gamma\delta}} \left\{ f_s(\vec{p};\vec{p}')\,\delta_{\alpha\beta}\delta_{\gamma\delta} + f_a(\vec{p};\vec{p}')\,\vec{\sigma}_{\alpha\beta}\cdot\vec{\sigma}_{\gamma\delta} \right\} \cdot \delta n_{p,\beta\alpha} \cdot \delta n_{p',\delta\gamma}$$

$f_{\alpha\beta,\gamma\delta}(p\,p')$ describes the forward scattering amplitude of quasi particles near the Fermi surface.



symmetry constraint:

orbital rotational symmetry $f(p,p')$ can only be a function of $\hat{p}\cdot\hat{p}'$,

spin-rotational symmetry: $\delta_{\alpha\beta}\delta_{\gamma\delta}$; $\vec{\sigma}_{\alpha\beta}\cdot\vec{\sigma}_{\gamma\delta}$

for particle p and p'

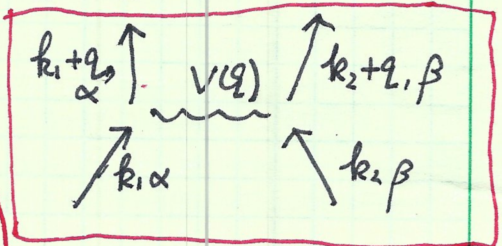
§ Calculation of Landau-interaction function at the tree level

Consider a spin-independent potential, in the 2nd quantization form, we have:

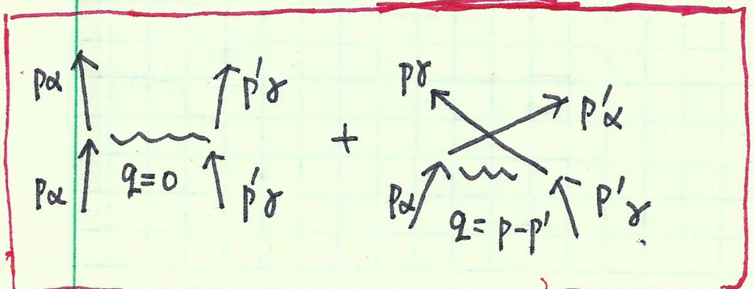
The interaction $H_{int} = \frac{1}{2}\int dr\, dr' \, \psi^\dagger_\alpha(r)\, \psi^\dagger_\beta(r')\, V(r-r')\, \psi_\beta(r')\, \psi_\alpha(r)$

$\xrightarrow{}$ Fourier transform $= \frac{1}{2V} \underset{V(q)}{\sum_{\alpha\beta}} C^\dagger_\alpha(k_1+q)\, C^\dagger_\beta(k_2-q)\, C_\beta(k_2)\, C_\alpha(k_1)$

where $V(q) = \int dr\, e^{-iqr}\, V(r)$ → interaction vertex



Fermi liquid interaction function corresponds to forward-scattering, i.e $q \to 0$.
However, due to indistinguishable processes, we have



$$f_{\alpha\beta,\gamma\delta}(\vec{p},\vec{p}') = V(q=0)\,\delta_{\alpha\beta}\,\delta_{\gamma\delta} - V(\vec{p}-\vec{p}')\,\delta_{\alpha\delta}\,\delta_{\beta\gamma}$$

using the identity

$$\frac{1}{2}\left[\vec{\sigma}_{\alpha\beta}\cdot\vec{\sigma}_{\gamma\delta} + \delta_{\alpha\beta}\delta_{\gamma\delta}\right] = \delta_{\alpha\delta}\delta_{\beta\gamma}$$

Ex: please prove it

Hint: express $\delta_{\alpha\delta}\delta_{\beta\gamma} = f_{\gamma\delta}\delta_{\alpha\beta} + \vec{g}_{\gamma\delta}\cdot\vec{\sigma}_{\alpha\beta}$ via trace

we have at the tree level

$$f_{\alpha\beta,\gamma\delta}(\vec{p},\vec{p}\,') = \left[V(0) - \frac{1}{2}V(p-p')\right]\delta_{\alpha\beta}\delta_{\gamma\delta} - \frac{1}{2}V(p-p')\,\vec{\sigma}_{\alpha\beta}\cdot\vec{\sigma}_{\gamma\delta}$$

Generally speaking, for system with spin conservation ($SU(2)$), the Landau interaction function can be represented as

$$\boxed{f_{\alpha\beta,\gamma\delta}(\vec{p},\vec{p}\,') = f^s(p,p')\,\delta_{\alpha\beta}\delta_{\gamma\delta} + f^a(p,p')\,\vec{\sigma}_{\alpha\beta}\cdot\vec{\sigma}_{\gamma\delta}.}$$

f^s and f^a describes the forward scattering amplitude which marks the fixed points of Fermi liquid in the RG language. f^s is in the density channel interaction, while f^a is in the spin channel.