Solution to HW 4 Phy 110 B

13.17 From text book Eq 13.34,

a)
$$H = T + U = \frac{1}{2} m \left[(c+1) \dot{z}^2 + (cz\dot{\phi})^2 \right] + mgz = \frac{1}{2m}\left[\frac{P_z^2}{(c^2+1)} + \frac{P_\phi^2}{c^2 z^2} \right] + mgz$$

$$\dot{z} = \frac{\partial H}{\partial P_z} = \frac{P_z}{m(c^2+1)} \qquad \dot{P}_z = -\frac{\partial H}{\partial z} = \frac{P_\phi^2}{mc^2 z^3} - mg$$

$$\dot{\phi} = \frac{\partial H}{\partial P_\phi} = \frac{P_\phi}{mc^2 z^2} \qquad \dot{P}_\phi = 0$$

$$\dot{z} = 0 \Rightarrow P_z = 0 = \frac{P_\phi^2}{mc^2 z_0^3} - mg = 0 \quad \Rightarrow \quad z_0 = \left(\frac{P_\phi^2}{m^2 c^2 g} \right)^{1/3}$$

b) Assume a small perturbation $z = z_0 + \epsilon \Rightarrow$

$$\ddot{\epsilon} = \ddot{z} = \frac{1}{m(c^2+1)} \dot{P}_z = \frac{1}{m(c^2+1)} \left[\frac{P_\phi^2}{mc^2 (z_0+\epsilon)^3} - mg \right]$$

$$= \frac{1}{m(c^2+1)} \left[\frac{P_\phi^2}{mc^2 z_0} \left[1 - \frac{3\epsilon}{z_0} \right] - mg \right] = -\frac{3 P_\phi^2 \epsilon}{m^2 c^2 (c^2+1) z_0^4}$$

this describes an oscillation around $\epsilon = 0$

c) the oscillation frequency $\omega^2 = \frac{3 P_\phi^2}{m^2 c^2 (c^2+1) z_0^4} = 3 \left(\frac{P_\phi}{mc^2 z_0^2} \right)^2 \frac{c^2}{c^2+1}$

$$= 3\dot{\phi}^2 \frac{tg^2\alpha}{1+tg^2\alpha} = 3\dot{\phi}^2 \sin^2\alpha$$

$$\omega = \sqrt{3}\dot{\phi} \sin\alpha$$

d) where $\omega = \dot{\phi} \Rightarrow \sin\alpha = \frac{1}{\sqrt{3}}$ or $\alpha \approx 35.3°$. Then the height z return to the initial value, for one circle. Therefore the orbital is closeel.

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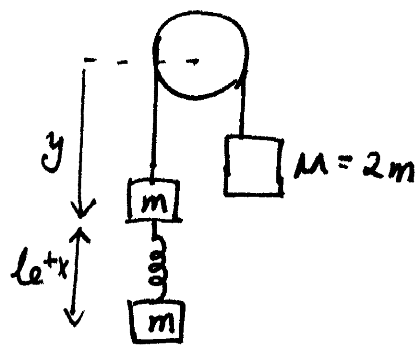
a) gravititational potential



$$U_g = (Mg - mg)\,y - mg(x+y) = -mgx$$

let l_0 denote the spring's natural length

$$k(l_e - l_0) = mg \Rightarrow l_e - l_0 = \frac{mg}{k}$$

$$\Rightarrow E_{spr} = \frac{1}{2}k(l_e + x - l_0)^2 = \frac{k}{2}\left(x + \frac{mg}{k}\right)^2 = \frac{1}{2}kx^2 + mgx + \text{Const}$$

$$\Rightarrow \quad U = U_g + E_{spr} = \frac{1}{2}kx^2$$

b) $$T = \frac{1}{2}m\dot{y}^2 + \frac{1}{2}M\dot{y}^2 + \frac{1}{2}m(\dot{x}+\dot{y})^2 = \frac{m}{2}\left[3\dot{y}^2 + (\dot{x}+\dot{y})^2\right]$$

$$P_x = \frac{\partial T}{\partial \dot{x}} = m(\dot{x}+\dot{y}), \quad P_y = \frac{\partial T}{\partial \dot{y}} = m(\dot{x}+4\dot{y})$$

$$\Rightarrow \quad \dot{x}+\dot{y} = \frac{P_x}{m}, \quad \dot{y} = \frac{1}{3m}(P_y - P_x)$$

$$\Rightarrow H = T + U = \frac{1}{2m}\left[\frac{(P_x - P_y)^2}{3} + P_x^2\right] + \frac{k}{2}x^2$$

c) $$\dot{x} = \frac{\partial H}{\partial P_x} = \frac{1}{3m}(4P_x - P_y) \qquad \dot{P}_x = -\frac{\partial H}{\partial x} = -kx$$

$$\dot{y} = \frac{\partial H}{\partial P_y} = \frac{1}{3m}(P_y - P_x), \qquad \dot{P}_y = -\frac{\partial H}{\partial y} = 0$$

initial $x(0) = x_0$, $y(0) = y_0$,

$$\dot{x}(0) = \dot{y}(0) = 0 \Rightarrow P_x(0) = P_y(0) = 0 \quad \Rightarrow P_y(t) = 0$$

$$\Rightarrow \quad \dot{x} = \frac{4}{3m}P_x, \quad \dot{P}_x = -kx \Rightarrow \ddot{x} = -\frac{4k}{3m}x \Rightarrow x = x_0\cos\omega t$$

with $\omega = \sqrt{\frac{4k}{3m}}$.

then $\dot{y} = \frac{-P_x}{3m} = -\frac{1}{3m}\frac{3m\dot{x}}{4} = -\frac{\dot{x}}{4} \Rightarrow y = -\frac{x}{4} + \text{const}$

plug in $y(0) = y_0$

and $x = x_0 \cos\omega t$

$\Rightarrow y = y_0 + \frac{1}{4} x_0 (1 - \cos\omega t)$.

PHYSICS 110B

HW #5

13.25

a) Given a system with one degree of freedom and Hamiltonian $\mathcal{H} = \mathcal{H}(q,p)$ which satisfies Hamilton's equations of motion:

$$\begin{cases} \dfrac{\partial \mathcal{H}}{\partial q} = -\dot{p} \\ \dfrac{\partial \mathcal{H}}{\partial p} = \dot{q} \end{cases}$$

We perform a canonical transformation to new coordinates $Q(q,p)$ and $P(q,p)$:

$$q = \sqrt{2P}\sin Q \quad ; \quad p = \sqrt{2P}\cos Q$$

Show that Hamilton's equations of motion are satisfied under the new coordinates:

$$\Rightarrow \frac{\partial \mathcal{H}}{\partial Q} = \underbrace{\frac{\partial \mathcal{H}}{\partial q}}_{-\dot{p}}\underbrace{\frac{\partial q}{\partial Q}}_{\sqrt{2P}\cos Q} + \underbrace{\frac{\partial \mathcal{H}}{\partial p}}_{\dot{q}}\underbrace{\frac{\partial p}{\partial Q}}_{-\sqrt{2P}\sin Q} = -\sqrt{2P}\left(\dot{q}\sin Q + \dot{p}\cos Q\right)$$

$$= \underbrace{-q\dot{q}}_{\dot{q}q - \frac{d}{dt}(q^2)} \underbrace{- p\dot{p}}_{\dot{p}p - \frac{d}{dt}(p^2)} = q\dot{q} + p\dot{p} - \frac{d}{dt}(q^2+p^2) \Rightarrow -q\dot{q} - p\dot{p} = -\frac{1}{2}\frac{d}{dt}\underbrace{(q^2+p^2)}_{2P}$$

, by product rule

$$\Rightarrow \boxed{\frac{\partial \mathcal{H}}{\partial Q} = -\dot{P}}$$

$$\Rightarrow \frac{\partial \mathcal{H}}{\partial P} = \underbrace{\frac{\partial \mathcal{H}}{\partial q}}_{-\dot{p}}\underbrace{\frac{\partial q}{\partial P}}_{\frac{\sin Q}{\sqrt{2P}}} + \underbrace{\frac{\partial \mathcal{H}}{\partial p}}_{\dot{q}}\underbrace{\frac{\partial p}{\partial P}}_{\frac{\cos Q}{\sqrt{2P}}} = \frac{1}{\sqrt{2P}}\left(\dot{q}\cos Q - \dot{p}\sin Q\right)$$

$$= \frac{1}{2P}\underbrace{(p\dot{q} - q\dot{p})}_{p^2\underbrace{\left(\frac{1}{p}\dot{q} - q\frac{\dot{p}}{p^2}\right)}_{\frac{d}{dt}\left(\frac{q}{p}\right)}} = \frac{1}{2P}p^2\underbrace{\frac{d}{dt}(\tan Q)}_{\dot{Q}\underbrace{\sec^2 Q}_{\frac{2P}{p^2}}} = \boxed{\dot{Q} = \frac{\partial \mathcal{H}}{\partial P}}$$

trick:

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b) Hamiltonian for 1D harmonic oscillator :

$$\mathcal{H}(q,p) = \frac{p^2}{2m} + \frac{1}{2}kq^2 = \boxed{\frac{1}{2}\left(q^2 + p^2\right)} \quad \text{for } m = k = 1$$

c) Rewrite the Hamiltonian in terms of Q, P by substitution :

$$\Rightarrow \mathcal{H}(Q,P) = \frac{1}{2}\left(2P\cos^2 Q + 2P\sin^2 Q\right) = P = \mathcal{H}(P)$$

$P = \mathcal{H}(P)$, thus Q is ignorable coordinate. P is apparently the Hamiltonian – the energy of the system.

d) Solve the Hamiltonian equation for $Q(t)$ by integration :

$$\Rightarrow \dot{Q} = \frac{\partial \mathcal{H}}{\partial P} = 1 \Rightarrow Q(t) = t - t_0 \text{, where } t_0 \equiv \text{const.}$$

$$\Rightarrow \dot{P} = \frac{\partial \mathcal{H}}{\partial Q} = 0 \Rightarrow P(t) = E \text{, the energy of the system (a const.)}$$

$$\therefore \begin{cases} q(t) = \sqrt{2E}\sin(t - t_0) \\ p(t) = \sqrt{2E}\cos(t - t_0) \end{cases}$$

which describes a SHO w/ energy E and frequency $\omega = \sqrt{\frac{k}{m}} = 1$

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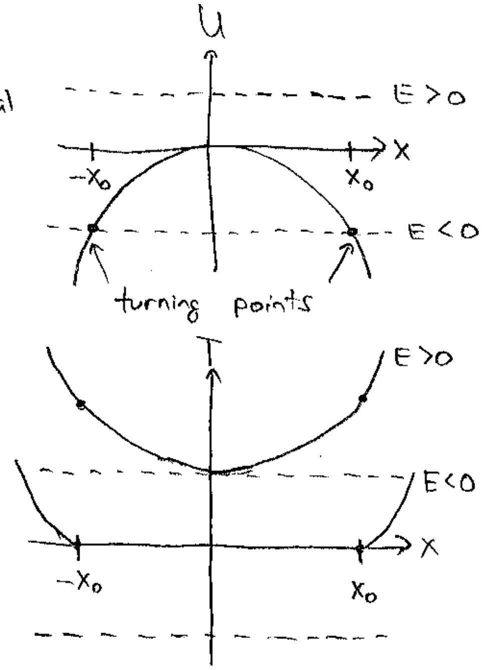
13.28

For mass m confined to the x axis and subject to force $F_x = kx$, $k>0$.

a) We write the potential energy $U(x)$:

$$\Rightarrow \quad F_x = -\frac{\partial U}{\partial x} = kx \quad \Rightarrow \quad \boxed{U(x) = -\frac{1}{2}kx^2}$$

where we set potential to zero at $x=0$



← obtained by $E = T + U$

Sketches of potential $U(x)$ and kinetic energy $T(x)$ illustrate possible motions of mass for following cases:

(i) case $E=0$; mass at $x=0$ for all time (unstable equilibrium)

(ii) case $E>0$; mass can go anywhere since $E = T+U$ and $T \geq 0$; mass can come in from the left, slowing down, and move out to the right, speeding up, or vice versa

(iii) case $E<0$; since $T \geq 0$, mass can't go between turning pts. but can come from left or right, slow to stopping at a turning point, then go back out speeding up

b) Write down Hamiltonian $\mathcal{H}(x,p) = T(p) + U(x) = \boxed{\frac{p^2}{2m} - \frac{1}{2}kx^2 = E, \text{ const.}}$

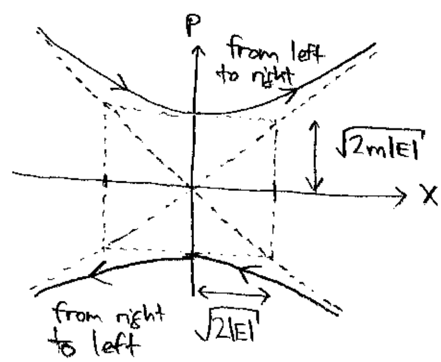
Describe possible phase-space orbits (case $E=0$ is trivial):

(ii) Case $E>0$:

$$\Rightarrow \boxed{\frac{1}{2m|E|}p^2 - \frac{k}{2|E|}x^2 = 1},$$

equation of a hyperbola

» All descriptions in previous part consistent with phase-space orbits here.



directions of phase flows are consistent since $p \gtrless 0$ corresponds to increasing / decreasing x

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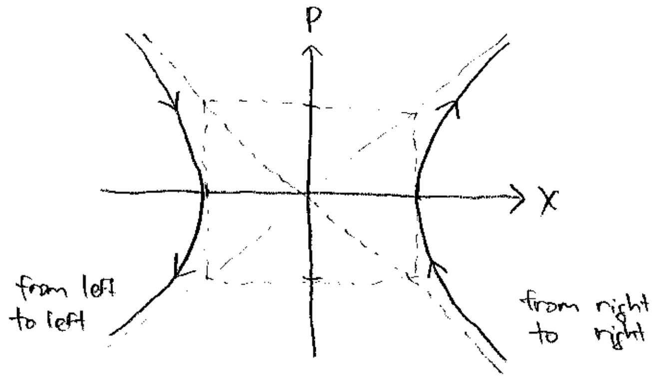
b) (iii) Case $E<0$:

$$\Rightarrow \boxed{\frac{-1}{2m|E|}p^2 + \frac{k}{2|E|}x^2 = 1},$$

another hyperbola



13.36 |

Prove Liouville's theorem, $\frac{dV}{dt}=0$ in the 2n-dimensional phase space of a system with n degrees of freedom:

Each phase point $\vec{z}$ moves through 2n-dimensional phase space in accordance with Hamiltonian's equations of motion:

$$\vec{z}=(\vec{q},\vec{p})=(q_1,\dots,q_n,p_1,\dots,p_n) \qquad ; \qquad \begin{cases} \dot{q}_i = \dfrac{\partial H}{\partial p_i} \\ \dot{p}_i = -\dfrac{\partial H}{\partial q_i} \end{cases}$$
$$=(z_1,\dots,z_n,z_{n+1},\dots,z_{2n})$$

Velocity of phase pt. is given by,

$$\vec{V}=\dot{\vec{z}}=(\underbrace{\dot{q}_1}_{\frac{\partial H}{\partial p_1}},\dots,\underbrace{\dot{q}_n}_{\frac{\partial H}{\partial p_n}},\underbrace{\dot{p}_1}_{-\frac{\partial H}{\partial q_1}},\dots,\underbrace{\dot{p}_n}_{-\frac{\partial H}{\partial q_n}})=(v_1,\dots,v_n,v_{n+1},\dots,v_{2n})$$

We can write the divergence of $\vec{v}$ (2n-dimensional):

$$\Rightarrow \vec{\nabla}\cdot\vec{v}=\sum_{i=1}^{2n}\frac{\partial v_i}{\partial z_i}=\frac{\partial^2 H}{\partial q_1\partial p_1}+\dots+\frac{\partial^2 H}{\partial q_n\partial p_n}-\frac{\partial^2 H}{\partial q_1\partial p_1}-\dots-\frac{\partial^2 H}{\partial q_n\partial p_n}=0$$

Thus, since $\vec{\nabla}\cdot\vec{v}=0$, the volume V enclosed by any arbitrary closed surface is a constant as the surface moves around phase space. ▨