Lect 7 Euler angles, spining top

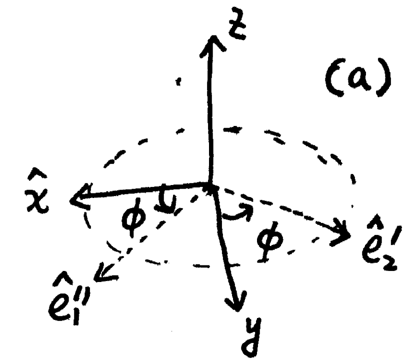
we first define a set of convenient coordinate (ϕ, θ, ψ) to describe rotation. as the following steps a) b), c)

a) Starting from aligning body axes and space axes together, rotate around $\hat{z}$ at the angle of ϕ. the body axis

from S: $(\hat{x}, \hat{y}, \hat{z}) \longrightarrow (\underbrace{\hat{e}_1'', \hat{e}_2', \hat{z}}_{\text{frame } B(a)})$

$$\boxed{\left.\begin{aligned} \hat{e}_1'' &= \cos\phi\, \hat{x} + \sin\phi\, \hat{y} \\ \hat{e}_2' &= -\sin\phi\, \hat{x} + \cos\phi\, \hat{y} \end{aligned}\right\}}$$

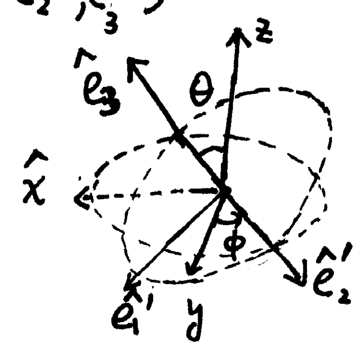
(a)

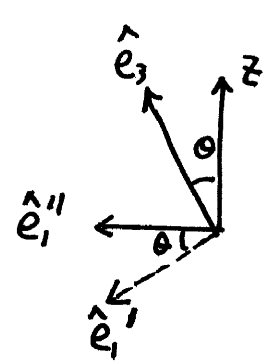


b) around $\hat{e}_2'$, rotate θ angle. Then the body axis

from $B(a)$ $(\hat{e}_1'', \hat{e}_2', \hat{z}) \longrightarrow B(b): [\hat{e}_1', \hat{e}_2', \hat{e}_3]$

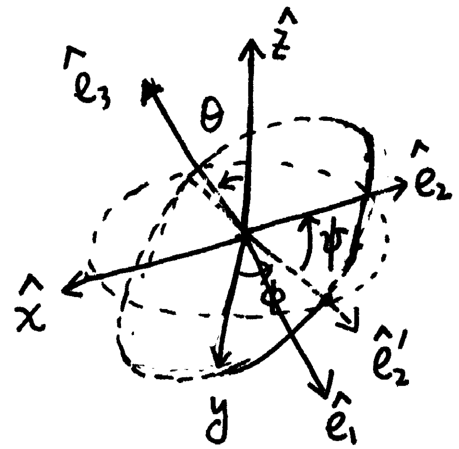
$$\boxed{\left.\begin{aligned} \hat{e}_3 &= \cos\theta\, \hat{z} + \sin\theta\, \hat{e}_1'' \\ &= \cos\theta\, \hat{z} + \sin\theta\cos\phi\, \hat{x} + \sin\theta\sin\phi\, \hat{y} \\ \hat{e}_1' &= -\sin\theta\, \hat{z} + \cos\theta\, \hat{e}_1'' \end{aligned}\right\}}$$



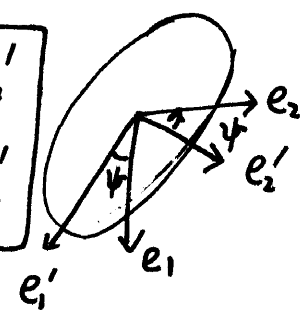


c) around $\hat{e}_3$, rotate ψ angle, then the body axis

$$B(b)\ [\hat{e}_1', \hat{e}_2', \hat{e}_3] \longrightarrow B(c)\ [\hat{e}_1, \hat{e}_2, \hat{e}_3]$$



$$\boxed{\begin{aligned}\hat{e}_1 &= \cos\psi\, \hat{e}_1' + \sin\psi\, \hat{e}_2' \\ \hat{e}_2 &= -\sin\psi\, \hat{e}_1' + \cos\psi\, \hat{e}_2'\end{aligned}}$$



Now let ϕ, θ. ψ change with time. $\dot{\psi}\hat{e}_3$, $\dot{\theta}\hat{e}_2'$, $\dot{\phi}\hat{z}$ describes the relative angular velocities of frames

$B(c) \to B(b) \to B(a) \to S$.

$$\boxed{\begin{aligned}&\phi \ \&\ \psi \text{ take } (0, 2\pi) \\ &\theta \text{ takes } (0, \pi)\end{aligned}}$$

$$\Rightarrow \vec{\omega} = \dot{\psi}\hat{e}_3 + \dot{\theta}\hat{e}_2' + \dot{\phi}\hat{z}$$

using the above transformation $\Rightarrow$

$$\hat{e}_2' = \cos\psi\, \hat{e}_2 + \sin\psi\, \hat{e}_1$$

$$\hat{z} = \cos\theta\, \hat{e}_3 - \sin\theta\, \hat{e}_1' = \cos\theta\, \hat{e}_3 - \sin\theta\,[\cos\psi\, \hat{e}_1 - \sin\psi\, \hat{e}_2]$$

$$= -\sin\theta\cos\psi\, \hat{e}_1 + \sin\theta\sin\psi\, \hat{e}_2 + \cos\theta\, \hat{e}_3$$

$$\Rightarrow \boxed{\begin{aligned}\vec{\omega} = {} & (-\dot{\phi}\sin\theta\cos\psi + \dot{\theta}\sin\psi)\, \hat{e}_1 + [\dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi]\, \hat{e}_2 \\ & + (\dot{\phi}\cos\theta + \dot{\psi})\, \hat{e}_3\end{aligned}}$$

Let us also find the transformation between e_i & xyz

$$\hat{e}_3 = \sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z}$$

$$\hat{e}_2 = -\sin\psi\,\hat{e}_1' + \cos\psi\,\hat{e}_2'$$

$$= -\sin\psi\,(-\sin\theta\,\hat{z} + \cos\theta\,\hat{e}_1'') + \cos\psi\,(-\sin\phi\,\hat{x} + \cos\phi\,\hat{y})$$

$$= -\sin\psi\,(-\sin\theta\,\hat{z} + \cos\theta\cos\phi\,\hat{x} + \cos\theta\sin\phi\,\hat{y}) + \cos\psi\,(-\sin\phi\,\hat{x} + \cos\phi\,\hat{y})$$

$$= (-\sin\psi\cos\theta\cos\phi - \cos\psi\sin\phi)\,\hat{x} + (-\sin\psi\cos\theta\sin\phi + \cos\psi\cos\phi)\,\hat{y}$$

$$+ \sin\psi\sin\theta\,\hat{z}$$

$$\hat{e}_1 = \cos\psi\,\hat{e}_1' + \sin\psi\,\hat{e}_2'$$

$$= \cos\psi\,(-\sin\theta\,\hat{z} + \cos\theta\cos\phi\,\hat{x} + \cos\theta\sin\phi\,\hat{y}) + \sin\psi\,(-\sin\phi\,\hat{x} + \cos\phi\,\hat{y})$$

$$= [\cos\psi\cos\theta\cos\phi - \sin\psi\sin\phi]\,\hat{x} + (\cos\psi\cos\theta\sin\phi + \sin\psi\cos\phi)\,\hat{y}$$

$$- \cos\psi\sin\theta\,\hat{z}$$

$$(e_1 e_2 e_3) = (x\,y\,z)\begin{bmatrix} \cos\psi\cos\theta\cos\phi - \sin\psi\sin\phi, & -\sin\psi\cos\theta\cos\phi - \cos\psi\sin\phi, & \sin\theta\cos\phi \\ \cos\psi\cos\theta\sin\phi + \sin\psi\cos\phi, & -\sin\psi\cos\theta\sin\phi + \cos\psi\cos\phi, & \sin\theta\sin\phi \\ -\cos\psi\sin\theta, & \sin\psi\sin\theta, & \cos\theta \end{bmatrix}$$

Let us also find the transformation between e_i & xyz

$$\hat{e}_3 = \sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z}$$

$$\hat{e}_2 = -\sin\psi\,\hat{e}_1' + \cos\psi\,\hat{e}_2'$$

$$= -\sin\psi(-\sin\theta\,\hat{z} + \cos\theta\,\hat{e}_1'') + \cos\psi(-\sin\phi\,\hat{x} + \cos\phi\,\hat{y})$$

$$= -\sin\psi(-\sin\theta\,\hat{z} + \cos\theta\cos\phi\,\hat{x} + \cos\theta\sin\phi\,\hat{y}) + \cos\psi(-\sin\phi\,\hat{x} + \cos\phi\,\hat{y})$$

$$= (-\sin\psi\cos\theta\cos\phi - \cos\psi\sin\phi)\,\hat{x} + (-\sin\psi\cos\theta\sin\phi + \cos\psi\cos\phi)\,\hat{y}$$

$$+ \sin\psi\sin\theta\,\hat{z}$$

$$\hat{e}_1 = \cos\psi\,\hat{e}_1' + \sin\psi\,\hat{e}_2'$$

$$= \cos\psi(-\sin\theta\,\hat{z} + \cos\theta\cos\phi\,\hat{x} + \cos\theta\sin\phi\,\hat{y}) + \sin\psi(-\sin\phi\,\hat{x} + \cos\phi\,\hat{y})$$

$$= [\cos\psi\cos\theta\cos\phi - \sin\psi\sin\phi]\,\hat{x} + (\cos\psi\cos\theta\sin\phi + \sin\psi\cos\phi)\,\hat{y}$$

$$- \cos\psi\sin\theta\,\hat{z}$$

$$(e_1 e_2 e_3) = (x\,y\,z)\begin{bmatrix} \cos\psi\cos\theta\cos\phi - \sin\psi\sin\phi, & -\sin\psi\cos\theta\cos\phi - \cos\psi\sin\phi, & \sin\theta\cos\phi \\ \cos\psi\cos\theta\sin\phi + \sin\psi\cos\phi, & -\sin\psi\cos\theta\sin\phi + \cos\psi\cos\phi, & \sin\theta\sin\phi \\ -\cos\psi\sin\theta, & \sin\psi\sin\theta, & \cos\theta \end{bmatrix}$$

$$\vec{L} = \lambda_1 \omega_1 \hat{e}_1 + \lambda_2 \omega_2 \hat{e}_2 + \lambda_3 \omega_3 \hat{e}_3$$

$$T = \frac{1}{2}\lambda_1 \omega_1^2 + \frac{1}{2}\lambda_2 \omega_2^2 + \frac{1}{2}\lambda_3 \omega_3^2 \xrightarrow{\text{Symmetric}} \lambda_1 = \lambda_2 = \lambda$$

$$\Rightarrow \boxed{T = \frac{\lambda}{2}\left[\dot{\phi}^2 \sin^2\theta + \dot{\theta}^2\right] + \frac{\lambda_3}{2}(\dot{\phi}\cos\theta + \dot{\psi})^2}$$

$$\omega_1 = -\dot{\phi}\sin\theta\cos\psi + \dot{\theta}\sin\psi$$

$$\omega_2 = \dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi$$

$$\omega_3 = \dot{\phi}\cos\theta + \dot{\psi}$$

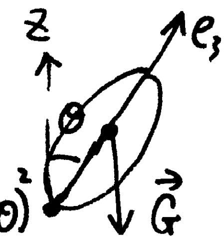
We can also express $\vec{\omega}$ in terms of $\hat{x}, \hat{y}, \hat{z}$

$$\dot{\phi}\hat{z}, \quad \dot{\theta}\hat{e}_2' = -[\dot{\theta}\sin\phi\,\hat{x} + \cos\phi\,\hat{y}]$$

$$\dot{\psi}\hat{e}_3 = \dot{\psi}\cos\theta\,\hat{z} + \dot{\psi}\sin\theta\cos\phi\,\hat{x} + \dot{\psi}\sin\theta\sin\phi\,\hat{y}$$

$$\Rightarrow \boxed{\begin{aligned}\vec{\omega} = &(-\dot{\theta}\sin\phi + \dot{\psi}\sin\theta\cos\phi)\,\hat{x} \\ &+ (\dot{\theta}\cos\phi + \dot{\psi}\sin\theta\sin\phi)\,\hat{y} \\ &+ \dot{\psi}\cos\theta\,\hat{z}\end{aligned}}$$

§ Lagrange's equation



$$\mathcal{L} = T - U = \frac{1}{2}\lambda(\dot\phi^2\sin^2\theta + \dot\theta^2) + \frac{\lambda_3}{2}(\dot\psi + \dot\phi\cos\theta)^2$$
$$- MgR\cos\theta$$

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot q}\right) = \frac{\partial\mathcal{L}}{\partial q} \Rightarrow \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot\phi}\right) = \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot\psi}\right) = 0$$

$$P_\phi = \frac{\partial\mathcal{L}}{\partial\dot\phi} = \lambda\dot\phi\sin^2\theta + \lambda_3(\dot\psi + \dot\phi\cos\theta)\cos\theta = \text{cost}$$

we have $\mathcal{L}_z = \vec{L}\cdot\hat{z} = \lambda\omega_1(-\cos\psi\sin\theta) + \lambda\omega_2\sin\psi\sin\theta$
$$+ \lambda_3\omega_3\cos\theta$$

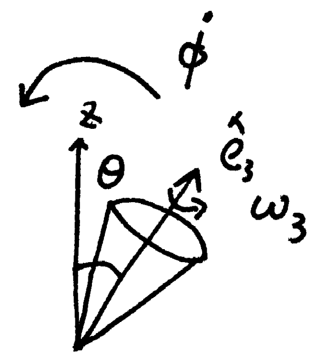
$$= \left[(-\dot\phi\sin\theta\cos\psi + \dot\theta\sin\psi)(\cos\psi) + (\dot\phi\sin\theta\sin\psi + \dot\theta\cos\psi)\sin\psi\right]\lambda\sin\theta$$
$$+ \underbrace{\lambda_3(\dot\phi\cos\theta + \dot\psi)}_{\mathcal{L}_3}\cos\theta$$
$$= [\lambda\dot\phi\sin^2\theta] + \mathcal{L}_3\cos\theta$$

$$\Rightarrow \boxed{P_\phi = \mathcal{L}_z = \text{const}} \leftarrow \lambda_3\omega_3$$

$$\boxed{P_\psi = \frac{\partial\mathcal{L}}{\partial\dot\psi} = \lambda_3(\dot\psi + \dot\phi\cos\theta) = \mathcal{L}_3 = \text{const.}}$$

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot\theta}\right) = \frac{\partial\mathcal{L}}{\partial\theta} \Rightarrow$$

$$\lambda \ddot{\theta} = \lambda \dot{\phi}^2 \sin\theta \cos\theta - \underbrace{\lambda_3 (\dot{\psi} + \dot{\phi} \cos\theta)}_{L_3 = \lambda_3 \omega_3} \dot{\phi} \sin\theta + MgR \sin\theta$$



§ Steady precession

if we fix $\theta = $ const.

From $L_z = \lambda \dot{\phi} \sin^2\theta + L_3 \cos\theta$

$$\Rightarrow \dot{\phi} = \frac{L_z - L_3 \cos\theta}{\lambda \sin^2\theta} = \Omega \leftarrow \text{const}$$

From $L_3 = \lambda_3 (\dot{\psi} + \dot{\phi} \cos\theta) = $ const $\Rightarrow$ $\dot{\psi}$ is also const

$$\Rightarrow \lambda \Omega^2 \cos\theta - \lambda_3 \omega_3 \Omega + MgR = 0$$

2 real roots at $\lambda_3^2 \omega_3^2 \geq 4 \lambda MgR \cos\theta$.

If ω_3 is large $\gg \Omega$ $\Rightarrow$ $\boxed{\Omega_1 \simeq \frac{MgR}{\lambda_3 \omega_3}}$ $\leftarrow$ precess mode due to gravity

$$\Omega_2 \simeq \frac{MgR}{\lambda \cos\theta} \cdot \frac{\lambda_3 \omega_3}{MgR} \approx \frac{\lambda_3 \omega_3}{\lambda_1 \cos\theta}$$

$$= \Omega_s$$

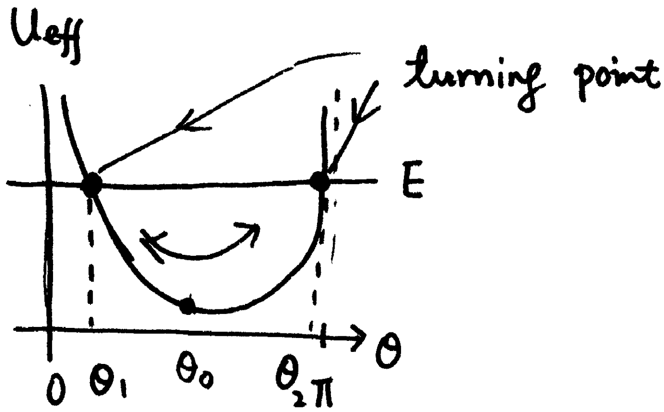
fast mode of free precession

§ $\boxed{\text{Nutation:}}$ we consider if θ varies, how to consider the mortion of a top.

$$T = \frac{1}{2}\lambda\left[(\dot{\phi}^2 \sin^2\theta) + (\dot{\theta})^2\right] + \frac{L_3^2}{2\lambda_3}$$

$$\dot{\phi} = \frac{L_z - L_3\cos\theta}{\lambda\sin^2\theta} \Rightarrow \dot{\phi}^2\sin^2\theta = \frac{(L_z - L_3\cos\theta)^2}{\lambda\sin^2\theta}$$

$$\Rightarrow E = T + MgR\cos\theta = \frac{1}{2}\lambda_1\dot{\theta}^2 + \frac{(L_z - L_3\cos\theta)^2}{2\lambda\sin^2\theta} + \frac{L_3^2}{2\lambda_3} + MgR\cos\theta$$

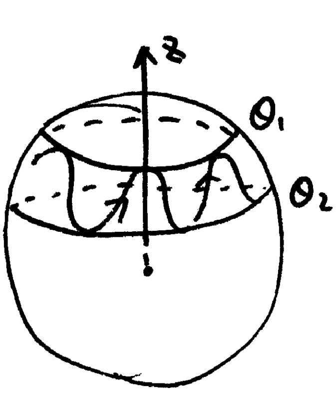


$$\dot{\phi} = \frac{L_z - L_3\cos\theta}{\lambda\sin^2\theta}$$

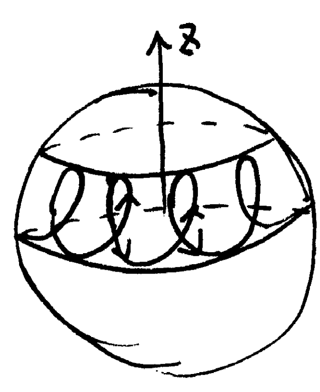
a) if $|L_z| > |L_3| \Rightarrow \dot{\phi} \neq 0$, top presess in a the same directi

b) if $L_z < L_3 \Rightarrow \dot{\phi} = 0$ at $\cos\theta_0 = \frac{L_z}{L_3}$. if θ_0 is between θ_1 & θ_2

$\Rightarrow \dot{\phi}$ change sign twice in each oscillati between θ_1 & θ_2.



(a)



(b)