Lect 18 Special Relativity (IV) E & M

§ introduction to tensors

① Vectors, tensors in 3D

$$\vec{a}' = R\vec{a} \quad \text{or} \quad a_i' = R_{ij} a_j,$$ where R is an orthogonal matrix satisfying $R^T R = 1$, i.e. $R_{ij} R_{ij'} = \delta_{jj'}$.

orthogonal matrix,

the inner product

$$\vec{a}\cdot\vec{b} = a^T b = a_i b_i$$

$$\vec{a}'\cdot\vec{b}' = a'^T b' = a^T R^T R b = a^T b = \vec{a}\cdot\vec{b}$$

② tensor $\quad T_{ij} \rightarrow T_{ij}' = R_{ii'} R_{jj'} T_{i'j'}$

each index transforms like 3-vector

or $\quad T' = R T R^T$

contraction $\quad b_i = \underbrace{T_{ij} a_j} \quad$ or $\quad b = T a$

only one index left ⇒ vector

$$b' = T' a' \rightarrow b' = R T R^T R a = R(Ta) = Rb.$$

§ 4-vector & tensors

covariance & contravariance vectors

$x^\mu = (x_1, x_2, x_3, ct)$ $\quad x_\mu = g_{\mu\nu} x^\nu = (x_1, x_2, x_3, -ct)$

$g^{\mu\nu} = g_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}$ $\quad x^\mu = g^{\mu\nu} x_\nu \leftarrow$ raise or lower index

Lorentz transformation $\Lambda^\mu{}_\nu \Rightarrow x'^\mu = \Lambda^\mu{}_\nu x^\nu$

Λ satisfies $\boxed{\Lambda^T g \Lambda = g, \text{ ie } \Lambda^{\mu'}{}_\mu g_{\mu'\mu'} \Lambda^{\nu'}{}_\nu = g_{\mu\nu}}$

$\Rightarrow x \cdot y = x^\mu g_{\mu\nu} y^\nu = x^T g y$

$x' \cdot y' = \Lambda^\mu{}_{\mu'} x^{\mu'} g_{\mu\nu} \Lambda^\nu{}_{\nu'} y^{\nu'} = x^{\mu'} \underbrace{\Lambda^\mu{}_{\mu'} g_{\mu\nu} \Lambda^\nu{}_{\nu'}}_{g_{\mu'\nu'}} y^{\nu'} = x \cdot y$

or $(\Lambda x)^T g (\Lambda y) = x^T \Lambda^T g \Lambda y = x^T g y = x \cdot y$

tensor $\quad T^{\mu\nu} \rightarrow T'^{\mu\nu} = \Lambda^\mu{}_{\mu'} \Lambda^\nu{}_{\nu'} T^{\mu'\nu'}$

$T_{\mu\nu} = g_{\mu\mu'} g_{\nu\nu'} T^{\mu'\nu'}$

Contract $\quad a = T \cdot x \Rightarrow T^{\mu\nu} x_\nu = a^\mu \leftarrow$ contravariance vector

$T^{\mu\nu} T'_{\mu\nu} \longrightarrow$ scalar $\quad$ Contract between contra- covariance indices!

§ 4-force 4-velocity

$$u^\mu = \frac{dx^\mu}{d\tau} \Rightarrow u^\mu = \left(\frac{\vec{v}}{\sqrt{1-\frac{v^2}{c^2}}}, \frac{c}{\sqrt{1-\frac{v^2}{c^2}}} \right)$$

$$K^\mu = \frac{dp^\mu}{d\tau} \Rightarrow K^\mu = \left(\frac{\vec{F}}{\sqrt{1-\frac{v^2}{c^2}}}, \frac{\vec{F}\cdot\vec{v}/c}{\sqrt{1-\frac{v^2}{c^2}}} \right) \leftarrow \frac{dp^0}{dt} = \vec{F}\cdot\vec{v}$$

$$\frac{dp^0}{c\,d\tau} = \gamma\left(\frac{\vec{F}\cdot\vec{v}}{c}\right)$$

we want to build up connection between K^μ and u^μ

define $K^\mu = F^{\mu\nu} u_\nu$, where $u_\nu = \left(\frac{\vec{v}}{\sqrt{1-\frac{v^2}{c^2}}}, \frac{-c}{\sqrt{1-\frac{v^2}{c^2}}} \right)$

Let us consider Lorentz force $\vec{F} = q(\vec{E} + \vec{v}\times\vec{B})$

$$K^1 = \gamma q(E_1 + v_2 B_3 - v_3 B_2) = q[B_3 u_2 - B_2 u_3 - (E_1/c)\, u_4]$$

$$K^2 = \gamma q(E_2 + v_3 B_1 - v_1 B_3) = q[-u_1 B_3 + u_3 B_1 - (E_2/c)\, u_4]$$

$$K^3 = \gamma q(E_3 + v_1 B_2 - v_2 B_1) = q[u_1 B_2 - u_2 B_1 - (E_3/c)\, u_4]$$

$$K^4 = \gamma q \frac{\vec{E}\cdot\vec{v}}{c} = \frac{q}{c}[E_1 u_1 + E_2 u_2 + E_3 u_3]$$

$$\rightarrow \begin{bmatrix} K^1 \\ K^2 \\ K^3 \\ K^4 \end{bmatrix} = q \begin{bmatrix} 0 & B_3 & -B_2 & -\frac{E_1}{c} \\ -B_3 & 0 & B_1 & -\frac{E_2}{c} \\ B_2 & -B_1 & 0 & -\frac{E_3}{c} \\ \frac{E_1}{c} & \frac{E_2}{c} & \frac{E_3}{c} & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}$$

$u^i = u_i$

$u^4 = -u_4$

$F^{\mu\nu}$ is the E-M tensor

$$F_{\mu\nu} = g_{\mu\mu'} g_{\nu\nu'} F^{\mu'\nu'} = \begin{bmatrix} 0 & +B_3 & -B_2 & \frac{E_1}{c} \\ -B_3 & 0 & B_1 & \frac{E_2}{c} \\ B_2 & -B_1 & 0 & \frac{E_3}{c} \\ -\frac{E_1}{c} & -\frac{E_2}{c} & -\frac{E_3}{c} & 0 \end{bmatrix}$$

$\Rightarrow$ define 4-vector

$$A^{\mu} = (\vec{A}, \ \phi/c), \quad A_{\mu} = (\vec{A}, \ -\phi/c)$$

$$\partial^{\mu} = (\vec{\partial}, \ -\tfrac{1}{c}\partial_t) \qquad \partial_{\mu} = (\vec{\partial}, \ \tfrac{1}{c}\partial_t)$$

Check $F^{\mu\nu} = \partial^{\mu}A^{\nu} - \partial^{\nu}A^{\mu}$, $F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\mu}A_{\nu}$

For $\mu, \nu = 1,2,3$

$$F^{ij} = \partial^i A^j - \partial^j A^i = \epsilon_{ijk} B^k$$

$\mu = 4$, $\nu = 1,2,3$

$$F^{4i} = -\frac{1}{c}\partial_t A^i - \partial^i \left(\frac{\phi}{c}\right) = \frac{E^i}{c}$$

$$F^{i4} = -\frac{E^i}{c}$$

Prove ∂^{μ} is contr covariance

$$\partial^{\mu} = \frac{\partial}{\partial x_{\mu}} \Rightarrow (\partial^{\mu})' = \frac{\partial}{\partial x'_{\mu}} = \frac{\partial}{\partial x_{\nu}} \frac{\partial x_{\nu}}{\partial x'_{\mu}} = \frac{\partial x_{\nu}}{\partial x'_{\mu}} \partial^{\nu}$$

$$x_{\nu} = g_{\nu\nu'} \Lambda^{-1\,\nu'}{}_{\lambda} \, g^{\lambda\mu} \, x'_{\mu}$$

$$\Rightarrow \quad \frac{\partial x_\nu}{\partial x'_\mu} = (g\Lambda^{-1}g)_\nu{}^\mu$$

we know $\Lambda^T g \Lambda = g$

$\Rightarrow \Lambda^T = g\Lambda^{-1}g$

$$= (\Lambda^T)_\nu{}^\mu = \Lambda^\mu{}_\nu$$

$$\Rightarrow (\partial^\mu)' = \Lambda^\mu{}_\nu \, \partial^\nu$$

$\Rightarrow$ A^μ 4-vector

∂^μ 4-vector

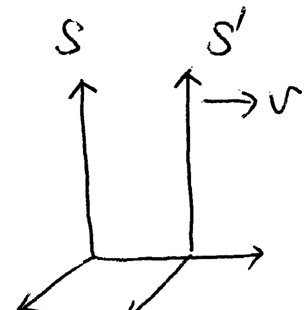
$\Rightarrow F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ tensor

§ transformation of E & B

Under Lorentz transformation

$$F'^{\mu\nu} = \Lambda^\mu{}_{\mu'} \Lambda^\nu{}_{\nu'} F^{\mu'\nu'}$$

For $\Lambda^\mu{}_{\mu'} = \begin{bmatrix} \gamma & 0 & 0 & -\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma\beta & 0 & 0 & \gamma \end{bmatrix}$



$$\boxed{\begin{array}{lll} E_1' = E_1 & E_2' = \gamma(E_2 - \beta c B_3) & E_3' = \gamma(E_3 + \beta c B_2) \\ B_1' = B_1 & B_2' = \gamma(B_2 + \beta E_3/c) & B_3' = \gamma(B_3 - \beta E_2/c) \end{array}}$$

Example: Fields of a long straight current

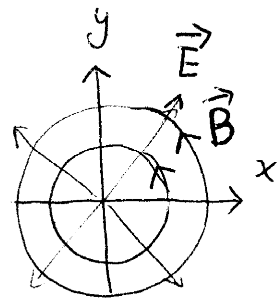
Frame S

$I = \lambda v$, $\vec{E} = \frac{2k\lambda}{\rho}\hat{\rho}$ where $k = \frac{1}{4\pi\varepsilon_0}$ $\leftarrow$ $E \cdot 2\pi\rho = 4\pi k \lambda$

$\uparrow$

line density

$\vec{B} = \frac{\mu_0}{2\pi}\frac{I}{\rho}\hat{\phi}$ $\longleftarrow$ $B \cdot 2\pi\rho = \mu_0 I$



in the comoving frame of S' at velocity v along $\hat{z}$-direction

$B' = 0$

due to length contraction $\Delta z' = \gamma \Delta z$

Charge is conserved $\Rightarrow$ $\lambda' = \frac{1}{\gamma}\lambda$

$\boxed{\begin{matrix} x' = x \\ y' = x \end{matrix}}$

$\vec{E}' = \frac{2k\lambda'}{\rho'^2}(x', y', 0) = \frac{2k\lambda}{\gamma\rho}\hat{\rho}$, ~~(x' y'~~ = X ~~don't~~

1 2 3
3 1 2

– For this transformation $\quad 1\,2\,3 \rightarrow 3\,1\,2$

$$E_3' = E_3 \qquad E_1' = \gamma(E_1 - \beta c B_2) \qquad E_2' = \gamma(E_2 + \beta c B_1)$$

$$B_3' = B_3 \qquad B_1' = \gamma(B_1 + \beta E_2/c) \qquad B_2' = \gamma(B_2 - \beta E_1/c)$$

$\downarrow$ inverse $\quad \beta \to -\beta$

$$E_3 = E_3' \qquad E_1 = \gamma(E_1' + \beta c B_2') \qquad E_2 = \gamma(E_2' - \beta c B_1')$$

$$B_3 = B_3' \qquad B_1 = \gamma(B_1' - \beta E_2'/c) \qquad B_2 = \gamma(B_2' + \beta E_1'/c)$$

– $\Rightarrow \quad \vec{E} = \gamma \vec{E}' = \dfrac{2k\lambda}{\rho}\,\hat{\rho}$ ✓

$$\vec{B} = +\beta\gamma\,\vec{E}' \times \hat{e}_\rho = \gamma\beta\,\frac{2k\lambda'}{c\rho^2}\,(-y, x, 0)$$

$$= \frac{\mu_0}{2\pi}\,\frac{\lambda v}{\rho}\,\hat{\phi}$$ ✓