Lect 14. Equation of motion of Solid & fluid

§1: Eq of motion

$$\rho dV \frac{\partial^2 \vec{u}}{\partial t^2} = \vec{F}_{vol} + \vec{F}_{sur}$$

$$\vec{F}_{vol} = \rho \vec{g} dV$$

$$\vec{F}_{sur} = \int \vec{\vec{\Sigma}} \cdot d\vec{A} \Rightarrow F_{sur,i} = \sum_j \int \sigma_{ij}\, dA_j = \int dV (\partial_j \sigma_{ij})$$

$$\vec{F}_{sur} = \underbrace{\vec{\nabla} \cdot \vec{\vec{\Sigma}}}_{dV} \Leftrightarrow F_{sur,i} = \partial_j \sigma_{ji}\, dV$$

$$\Rightarrow \rho \frac{\partial^2 \vec{u}}{\partial t} = \rho \vec{g} + \vec{\nabla} \cdot \vec{\vec{\Sigma}}$$

$\vec{\vec{\Sigma}} = (\alpha - \beta) \vec{\vec{I}} + \beta \vec{\vec{E}}$ or in each compnets $\sigma_{ij} = (\alpha - \beta) e \delta_{ij} + \beta \epsilon_{ij}$

$$e = \frac{1}{3} \epsilon_{ii} = \frac{1}{3} \vec{\nabla} \cdot \vec{u}$$

$$\epsilon_{ij} = \frac{1}{2} (\partial_i u_j + \partial_j u_i)$$

$$\Rightarrow \sigma_{ij} = \frac{1}{3} (\alpha - \beta) \delta_{ij} (\nabla \cdot \vec{u}) + \frac{1}{2} \beta (\partial_i u_j + \partial_j u_i)$$

$$(\vec{\nabla} \cdot \vec{\vec{\Sigma}})_i = \partial_j \sigma_{ji} = \frac{1}{3} (\alpha - \beta) \delta_{ji}\, \partial_i (\nabla \cdot \vec{u}) + \frac{1}{2} \beta [\underbrace{\partial_j \partial_i u_j}_{\partial_i (\nabla \cdot \vec{u})} + \partial_j^2 u_i]$$

$$(\vec{\nabla} \cdot \vec{\vec{\Sigma}}) = \left(\frac{\alpha}{3} + \frac{\beta}{6}\right) \vec{\nabla} (\vec{\nabla} \cdot \vec{u}) + \frac{\beta}{2} \nabla^2 \vec{u}$$

$$= \left(BM + \frac{SM}{3}\right) \vec{\nabla} (\vec{\nabla} \cdot \vec{u}) + SM\, \nabla^2 \vec{u}$$

$$\Rightarrow \boxed{\rho \frac{\partial^2 \vec{u}}{\partial t^2} = \rho \vec{g} + \left(BM + \frac{SM}{3}\right) \vec{\nabla} (\vec{\nabla} \cdot \vec{u}) + SM\, \nabla^2 \vec{u}}$$

§2. Longitudinal v.s transverse wave

① set $g=0$.

② conside a longitudinal disturbunce, $\vec{u} = [u_x(x,t), 0, 0]$

$$\vec{\nabla}\cdot\vec{u} = \partial_x u_x \Rightarrow \vec{\nabla}(\vec{\nabla}\cdot\vec{u}) = \partial_x^2 u_x \hat{e}_x, \quad \nabla^2\vec{u} = \partial_x^2 u_x \hat{e}_x$$

$$\Rightarrow \rho \frac{\partial^2 u_x}{\partial t^2} = \left(BM + \frac{4}{3}SM\right)\left(\frac{\partial^2 u_x}{\partial x^2}\right) \Rightarrow C_L = \sqrt{\frac{BM + \frac{4}{3}SM}{\rho}}$$

③ consider a transverse disturbance $\vec{u} = [0, u_y(x,t), 0]$

$$\vec{\nabla}\cdot\vec{u} = 0$$

$$\Rightarrow \rho \frac{\partial^2 u_y}{\partial t^2} = SM \nabla^2 u_y \Rightarrow C_T = \sqrt{\frac{SM}{\rho}}$$

☆ Longitudinal wave is faster than transverse wave

see example 7.2.2 $\quad C_L \approx 5.3$ km/s, $\quad C_T \approx 3$ km/s

in liquid, gas, $SM = 0 \Rightarrow$ <u>only longitudinal wave</u> !

§3 fluid:

material description: $\vec{r} \to \vec{r} + \vec{u}(r,t)$. follow each pieces of material.

spatial description: convenient for fluids.

look at $v(r,t)$, $\rho(r,t)$ at each point. (In fluid, we don't have well-defined equillibrium positions).

material derivative:

$$\frac{d\rho(r,t)}{dt} = \frac{\partial \rho}{\partial t} + \nabla\rho \cdot \frac{\partial \vec{r}}{\partial t} = \frac{\partial \rho}{\partial t} + \vec{v}\cdot\nabla\rho$$

$$\frac{dv(r,t)}{dt} = \frac{\partial v}{\partial t} + \vec{v}\cdot\nabla\vec{v}$$

Eq of motion for an inviscid fluid.

$$\rho\, dV \frac{d\vec{v}}{dt} = \vec{F} = \vec{F}_{vol} + \vec{F}_{sur}$$

$$\vec{F}_{sur} = \vec{\nabla}\cdot\overleftrightarrow{\Sigma}\, dV .$$

In fluids. $\overleftrightarrow{\Sigma} = -p\overleftrightarrow{I} \Rightarrow \partial_j \sigma_{ji} = \partial_j (-p)\delta_{ij} = -\partial_i p$

$$\Rightarrow \vec{F}_{sur} = -\vec{\nabla}\cdot p$$

$$\Rightarrow \boxed{\rho\frac{d\vec{v}}{dt} = \rho\vec{g} - \nabla p}$$

Bernouli's theorem:

Let's conside steady, impressible fluid . motion.

$$\frac{\partial \rho}{\partial t}, \frac{\partial p}{\partial t}, \frac{\partial v}{\partial t}, = 0. \qquad \frac{d\rho}{dt} = 0$$

$$\rho \frac{d\vec{v}}{dt} = \rho\vec{g} - \nabla p \Rightarrow \rho\vec{v}\cdot\frac{d\vec{v}}{dt} + \rho\vec{v}\cdot\nabla(gz) + \vec{v}\cdot\nabla p = 0$$

$$\vec{g} = -\nabla(gz)$$

$$\vec{v}\cdot\nabla f = \frac{df}{dt} - \frac{\partial f}{\partial t} \quad : \text{ for steady flow } \frac{\partial f}{\partial t} = 0 \Rightarrow \vec{v}\cdot\nabla f = \frac{df}{dt}$$

$$\Rightarrow \frac{1}{2}\rho\frac{dv^2}{dt} + \rho\frac{d(gz)}{dt} + \frac{dp}{dt} = 0$$

const if follow the flow line.

$$\Rightarrow \frac{d\rho}{dt} = 0 \Rightarrow \frac{1}{2}\rho\frac{dv^2}{dt} = \frac{1}{2}\frac{d\rho v^2}{dt} \Rightarrow \boxed{\frac{d}{dt}\left[\frac{\rho}{2}v^2 + \rho g z + p\right] = 0}$$

$$\rho\frac{d(gz)}{dt} = d\left(\frac{(\rho g z)}{dt}\right)$$

dynamic pressure.

Continuity Eq

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\vec{v}) = 0$$

$$\frac{\partial \rho}{\partial t} = \frac{d\rho}{dt} - \vec{v}\,\nabla\rho$$

$$\Rightarrow \boxed{\frac{d\rho}{dt} - \vec{v}\,\nabla\rho + \nabla(\rho\vec{v}) = \frac{d\rho}{dt} + \rho\nabla\vec{v} = 0}$$

§ waves in a fluid

Let us consider a disturbance in fluids

$$P = P_0 + P'(r,t) \quad \& \quad \rho = \rho_0 + \rho'(r,t)$$

From $\rho \frac{d\vec{v}}{dt} = \rho \vec{g} - \nabla P \quad \Rightarrow \quad (\rho_0 + \rho')\left(\frac{\partial \vec{v}}{\partial t} + \vec{v}\cdot\nabla\vec{v}\right) = (\rho_0 + \rho')g - \nabla(P_0 + P'$

at equilibrium $\rho_0 \vec{g} - \nabla P_0 = 0$

Neglecting second order terms $\vec{v}\cdot\nabla\vec{v}$, $\rho'\left(\frac{\partial \vec{v}}{\partial t} + \vec{v}\cdot\nabla\vec{v}\right)$

$\Rightarrow \quad \rho_0 \frac{\partial \vec{v}}{\partial t} = \rho' g - \nabla P'$. $\xrightarrow[\text{gravity}]{\text{neglect}}$ $\underline{\rho_0 \frac{\partial \vec{v}}{\partial t} = -\nabla P'}$

From continuity $\frac{\partial \rho}{\partial t} + \nabla(\rho \vec{v}) = 0 \Rightarrow \frac{\partial}{\partial t}(\rho_0 + \rho') + \nabla(\rho_0 + \rho')\vec{v}' + (\rho_0 + \rho')\nabla\vec{v}' = 0$

neglecting second order $\Rightarrow \frac{\partial}{\partial t}\rho' = -\rho_0 \nabla\vec{v} - \vec{\nabla}\rho_0 \cdot \vec{v}$

↑ due to gravity effect is small

$\Rightarrow \quad \underline{\frac{\partial}{\partial t}\rho' = -\rho_0 \nabla\vec{v}}$

$dp = BM\left(-\frac{dV}{V}\right)$ and $d(\rho V) = 0 \Rightarrow \rho dV + V d\rho = 0$

$$-\frac{dV}{V} = \frac{d\rho}{\rho}$$

$\Rightarrow dp = BM \frac{d\rho}{\rho}$ or $\boxed{p' = BM \cdot \frac{\rho'}{\rho_0}}$

$$\Rightarrow \frac{\partial p'}{\partial t} = \frac{BM}{\rho_0}\frac{\partial \rho'}{\partial t} = -BM \nabla \vec{v}$$

$$\frac{\partial^2}{\partial t^2} p' = -BM \nabla \dot{\vec{v}} = -BM \nabla \frac{\partial}{\partial t}\vec{v} = \frac{BM}{\rho_0}\nabla^2 p'$$

$$\Rightarrow \boxed{c = \sqrt{\frac{BM}{\rho_0}}}$$

§ Longitudinal nature of pressure waves in fluids.

$$p' = f(\vec{k}\cdot\vec{r} - ct)$$

$$\frac{\partial \vec{v}}{\partial t} = -\frac{1}{\rho_0}\nabla p' = -\frac{\vec{k}}{\rho_0} f'(\vec{k}\cdot\vec{r} - ct)$$

← derivative respect to its $\vec{k}\cdot\vec{r} - ct$

$$\vec{v} = -\frac{\vec{k}}{\rho_0}\int f'(\vec{k}\cdot\vec{r} - ct)\,dt = \frac{\vec{k}}{c\rho_0}\int f'(\vec{k}\vec{r} - ct)\,d(ct)$$

$$= \frac{\vec{k}}{c\rho_0} f(\vec{k}\cdot\vec{r} - ct) = \frac{p'\vec{k}}{c\rho_0} \Rightarrow \boxed{\vec{v} \parallel \vec{k}}$$