Take home Midterm - Phy110B, due on class on Feb 12 Friday

Please write down a sentence to express that you have done the midterm independently without discussion with others, and sign your name.

Problem 1. Projectile motions (15 points).

Consider projectile motions at the latitude $\theta°$ N (the northern hemisphere) in the convention that the equator is at $0°$. The local gravity acceleration is g and vertically down. You can neglect the small deviation of the gravity due to the centrifugal acceleration from the earth spinning. The air resistance can also be neglected. You need to consider the deviation of the projectile motion due to the Coriolis force for the following two cases.

The earth spinning angular velocity is Ω. For the following two questions, you only need to keep the linear order of Ω.

1) The projectile starts to move vertically up from the ground, and reaches the height of h, and then lands on the ground. What is deviation of the landing point from its starting point? Please give both the direction and the distance.

2) Now the projectile is a cannon ball. It is shot towards east with an initial speed v_0 and an angle α with respect to the ground. Only consider the transverse deviation away from the east-west direction. What is the deviation of the landing point of the cannon ball? Give the direction (north or south) and the distance. (You do not need to calculate the longitudinal deviation along the east-west direction.)

Problem 2. Moment of inertial tensor (8 points)

1) Consider a rigid body with the mass density distribution $\rho(\vec{r})$. The moment of inertial tensor in the Lab frame with fixed x, y and z axis is defined as

$$I_{ij} = \int d\vec{r}\rho(\vec{r})(r^2\delta_{ij} - r_i r_j), \tag{1}$$

where $i, j = x, y$ and z, respectively. Let us perform a rotation T to the rigid body such that $\vec{r}$ is transformed into $\vec{r}'$ with the following relation

$$r_i' = T_{ij}r_j, \tag{2}$$

where T_{ij} is a 3×3 orthogonal matrix.

How does the moment of inertial tensor transform under this rotation? Prove your conclusion by applying the coordinate transformation Eq. 2 to Eq. 1. (Hint: You need to use some properties of the orthogonal matrix.)

2) Consider a cube with a uniform density and total mass M. Its edge length is a. Let us put the center of the cube at the origin but otherwise leave the cube configuration arbitrary. Prove that the momentum of inertial tensor I_{ij} is the same regardless of the configuration of the cube. What is it?

Problem 3. Nutation of the symmetric top (12 points).

Please read from Chapter 10.7 to Chapter 10.10 carefully before doing this problem. We follow the same notation therein. In class, we have derived that the effective potential of a spinning symmetric top in the gravity field as

$$U_{eff}(\theta) = \frac{(L_z - L_3 \cos\theta)^2}{2\lambda \sin^2\theta} + \frac{L_3^2}{2\lambda_3} + MgR\cos\theta, \tag{3}$$

where L_z and L_3 are constants of motion determined by the initial condition; λ_3 is the moment of inertial eigenvalue around the symmetry axis $\hat{e}_3$, and $\lambda = \lambda_1 = \lambda_2$ are the other two eigenvalues. You can directly use Eq. 3 for the following questions.

Consider the initial condition at $t = 0$: The top is set at $\theta|_{t=0} = \theta_1$ and it is only spinning around its symmetry axis $\hat{e}_3$, i.e., $\dot{\psi}|_{t=0} = \omega$, $\dot{\phi}|_{t=0} = \dot{\theta}|_{t=0} = 0$.

1) What are the values of L_3 and L_z? Explain why θ_1 is one of the turning points of the nutation.

Denote that θ_2 is the other turning point. We define $x = \cos\theta_1 - \cos\theta_2$ as the range of nutation, and calculate x below. Define $p = \lambda_3^2 \omega^2/(2\lambda MgR)$ and consider the fast spinning limit of $p \gg 1$.

2) Derive the expression of x correct to the leading order of $1/p$.

3) Consider a realistic situation that the top spinning slows down by friction. Will the nutation range x increase or decrease? (No calculation is needed here, but you need to give reasoning.)

4) In class, we derived the relation $\dot{\phi} = (L_z - L_3 \cos\theta)/(\lambda \sin^2\theta)$. Based on this equation and your result in 2), estimate the average value of the precession angular velocity $\bar{\dot{\phi}}$ in the limit of $p \gg 1$, and compare with the result you already got in the lecture.