Lect 7: Applications of energy

§ 1D motion : F_x

If F_x is only coordinate-dependent, then F_x is conservative. This is because any closed loop in 1D has to come back along the same path

$$\boxed{\int_1^2 dx\, F_x + \int_2^1 dx\, F_x = 0}$$

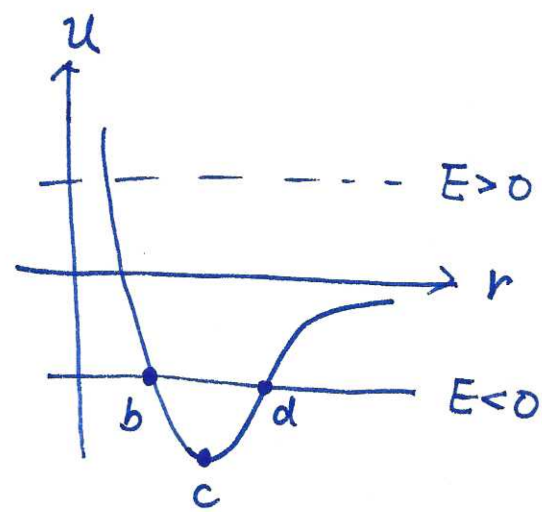
Then the potential energy $U(x)$ can be simply integrated as

$$\boxed{U(x) = -\int_{x_0}^{x} F_x(x')\,dx'}$$

x_0 can be any point

$U(x)$ with different x_0 is up to a constant.

potential energy for a diatomic molecule.



① $E<0$, at b and d ⇒ $T=0$, turning points. At c, $\frac{\partial U}{\partial r} = 0$, $\frac{\partial^2 U}{\partial r^2} > 0$.

c is equilibrium point

$E<0$ — bound states

② $E>0$ — scattering states

we can formally complete solution of motion in 1D

$$T = \frac{1}{2} m \dot{x}^2 = E - U(x) \Rightarrow \dot{x}(x) = \pm \sqrt{\frac{2}{m}} \sqrt{E - U(x)}$$

The direction of $\dot{x}(x)$ can be either right / left mover.

we also have $\dot{x} = \frac{dx}{dt} \Rightarrow dt = \frac{dx}{\dot{x}(x)}$

$$\Rightarrow \int_{t_i}^{t_f} dt = \boxed{\int_{x_i}^{x_f} \frac{dx}{\dot{x}(x)} = t_f - t_i}$$

Suppose $\dot{x}$ is positive, we have $t_f - t_i = \sqrt{\frac{m}{2}} \int_{x_0}^{x} \frac{dx'}{\sqrt{E - U(x')}}$.

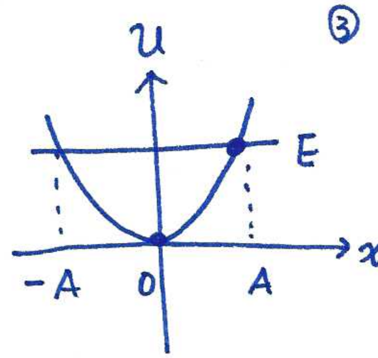
$\dot{x}$ can change directions at turning points, and we can treat by dividing the motion into different regions. In each region, $\dot{x}$'s direction is fixed, and we add the time of each reigon together.

Example: free fall: $U(z) = -mgz$ and $\begin{cases} E = 0 \\ \text{at } z_{in} = 0 \\ v_{in} = 0 \end{cases}$

$$\Rightarrow \dot{z}(z) = \sqrt{\frac{2}{m}} \sqrt{E - U(z)} = \sqrt{2gz}$$

$$t = \int_0^z \frac{dz'}{\dot{z}(z')} = \int_0^z \frac{dz'}{\sqrt{2gz'}} = \sqrt{\frac{2z}{g}} \Rightarrow z = \frac{1}{2} g t^2$$

2: harmonic osceillater



$U = \frac{1}{2} k x^2$ with energy E.

The turning points at $\pm A$, with $\frac{1}{2} k A^2 = E$.

conside at $\begin{cases} t_{in} = 0 \\ x_0 = A \end{cases}$ and at $\begin{cases} t_f = T/4 \\ x_f = 0 \end{cases}$

we have $\dot{x}(x) = -\sqrt{\frac{2}{m}} \left(E - \frac{1}{2} k x^2\right)^{1/2}$

$$\Rightarrow T/4 = +\int_A^0 \frac{dx}{\dot{x}} = \sqrt{\frac{m}{2}} \int_0^A dx \frac{1}{(E - \frac{1}{2} k x^2)^{1/2}}$$

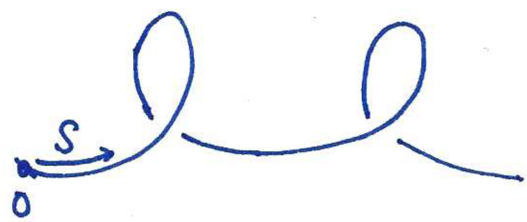
$$= \sqrt{\frac{m}{2}} \left(\frac{k}{2}\right)^{-1/2} \cdot \int_0^A dx \frac{1}{A\left(1 - \left(\frac{x}{A}\right)^2\right)^{1/2}}$$

$$= \sqrt{\frac{m}{k}} \int_0^1 dy \frac{1}{(1-y^2)^{1/2}} = \omega_0^{-1} \arcsin y \Big|_0^1 = \frac{\pi}{2\omega_0}$$

$$\Rightarrow T = \frac{2\pi}{\omega_0} \quad \text{where } \omega_0 = \sqrt{k/m}.$$

§. effective 1D systems

1. motion along curves



we use the arc length S as coordinate.

Since the motion is constrained along the curve, the direction of the velocity is along the tangential direction $\vec{e}_t$, and the speed is simply $\dot{S}$. $\Rightarrow \vec{v}(S) = \dot{S}\, \vec{e}_t(S)$

$$\Rightarrow \vec{a}(s) = \ddot{s}\,\vec{e}_t(s) + \dot{s}\frac{d}{dt}\vec{e}_t(s) = \ddot{s}\,\vec{e}_t(s) + (\dot{s})^2 \frac{d\vec{e}_t(s)}{ds}$$

$$\vec{e}_t \cdot \frac{d\vec{e}_t(s)}{ds} = \frac{d(\vec{e}_t\cdot\vec{e}_t)}{ds} = 0 \Rightarrow \frac{d\vec{e}_t}{ds} \perp \vec{e}_t(s)$$

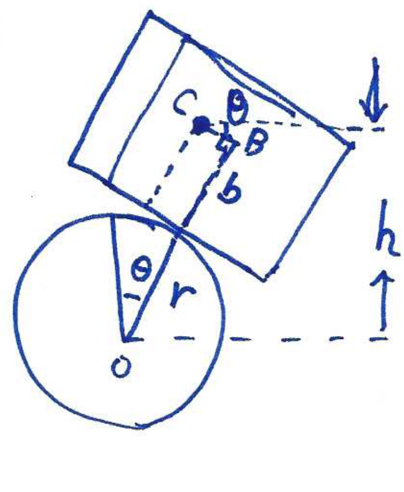
$$\Rightarrow \boxed{F_{tang} = m\ddot{s}}$$

The normal force don't do work.

Thus as long as, we take s as the coordinate, the 1D motion along curves is reduced to the usual 1D motion. We can define

$$\boxed{F_{tang} = -dU(s)/ds \quad \text{and} \quad \frac{1}{2}m\dot{s}^2 + U(s) = E.}$$

2: Ex: Stability of a cub balanced on a cylinder



The mass center of the cube is at C, the cube edge length is 2b. The cube can roll on the surface of the cylinder without slide.

Then CB equals the distance of roll $r\theta$

$\Rightarrow$ the height of C relative to "O" is

$$(r+b)\cos\theta + r\theta\sin\theta = h$$

$$\Rightarrow U(\theta) = mgh = mg\left[(r+b)\cos\theta + r\theta\sin\theta\right]$$

$$\frac{\partial \mathcal{U}}{\partial \theta} = mg\left[-(r+b)\sin\theta + r\sin\theta + r\theta\cos\theta\right] = mg\left[r\theta\cos\theta - b\sin\theta\right]$$

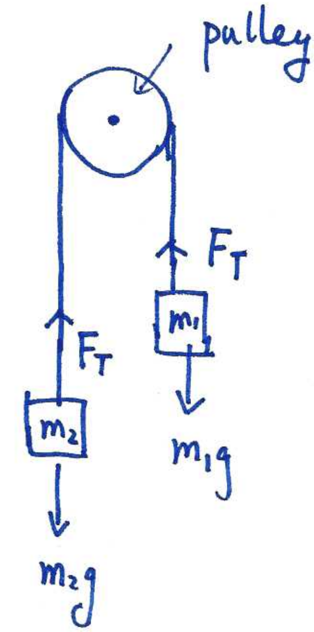
at $\theta = 0$, $\frac{\partial \mathcal{U}}{\partial \theta} = 0$. In order to check if it is a stable equilibri

$$\Rightarrow \frac{\partial^2 \mathcal{U}}{\partial \theta^2} = mg\left[r\cos\theta - r\theta\sin\theta - b\cos\theta\right]\Big|_{\theta=0} = mg(r-b)$$

$\Rightarrow$ at $b < r$ (cube is smaller), $\Rightarrow \frac{\partial^2 \mathcal{U}}{\partial \theta^2} > 0$, it's stable

$b > r$ cube is large $\Rightarrow \frac{\partial^2 \mathcal{U}}{\partial \theta^2} < 0$, it's unstable.

3: Atwood machine



Two mass-points suspended by a massless inextensible string

$$\Delta T_1 + \Delta \mathcal{U}_1 = W_1^{ten}$$

$$\Delta T_2 + \Delta \mathcal{U}_2 = W_2^{ten}$$

$\mathcal{U}_{1,2}$ only count gravity potential

The tensions on m_1 and m_2 are the same.

$dS_1 + dS_2 = 0$ — string length is fixed

$$\Rightarrow W_1^{ten} + W_2^{ten} = \int dS_1\, W_1^{ten} + \int dS_2\, W_2^{ten} = 0$$

$$\Rightarrow \Delta(T_1 + T_2 + \mathcal{U}_1 + \mathcal{U}_2) = 0 \Rightarrow E = T_1 + T_2 + \mathcal{U}_1 + \mathcal{U}_2$$

In general, if a system contains several particles constrained in certain way, and if the constraining forces do not do work on the system as a whole, then we can neglect them in writing down the conserved total energy

$$E = \sum_{\alpha=1}^{N} (T_\alpha + U_\alpha).$$