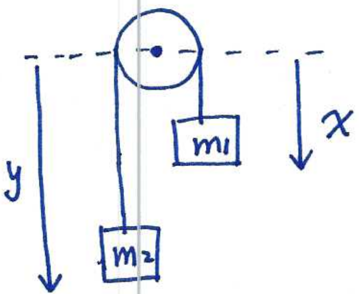
Lect 16: More applications of Lagrangian

1 Atwood's machine



$$x + y = \text{const} \Rightarrow \dot{x} = -\dot{y}$$

$$T = \frac{1}{2} m_1 \dot{x}^2 + \frac{1}{2} m_2 \dot{y}^2 = \frac{1}{2}(m_1+m_2)\dot{x}^2$$

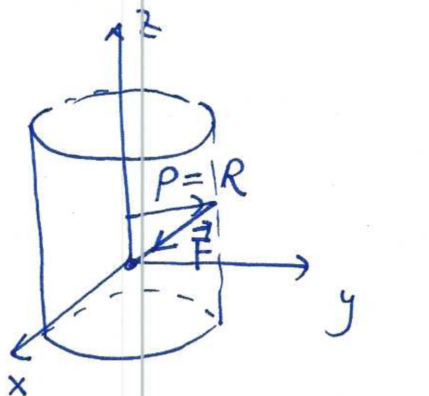
$$U = -m_1 g x - m_2 g y = -(m_1 - m_2) g x + \text{const}$$

$$\mathcal{L} = T - U = \frac{1}{2}(m_1+m_2)\dot{x}^2 + (m_1 - m_2) g x$$

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) \Rightarrow (m_1 - m_2) g = (m_1+m_2)\ddot{x} \Rightarrow \boxed{\ddot{x} = \frac{m_1 - m_2}{m_1 + m_2} g}$$

No need to involve the constraint force along the rope.

2 particile confined on a cylinder



$$\vec{v} = \dot{\rho}\,\hat{e}_\rho + \rho\dot{\theta}\,\hat{e}_\theta + \dot{z}\,\hat{e}_z$$

$$\rho = R = \text{cst}$$

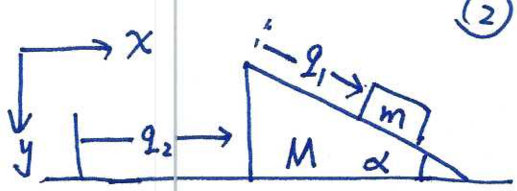
$$\Rightarrow \vec{v} = R\dot{\theta}\,\hat{e}_\theta + \dot{z}\,\hat{e}_z$$

$$T = \frac{1}{2} m\left(R^2\dot{\theta}^2 + \dot{z}^2\right). \qquad U = \frac{1}{2} k\left(R^2 + z^2\right) \leftarrow \vec{F} = -k\vec{r}$$

$$\Rightarrow \mathcal{L} = \frac{1}{2} m\left(R^2\dot{\theta}^2 + \dot{z}^2\right) - \frac{1}{2} k z^2 + \text{const}$$

$$\frac{\partial \mathcal{L}}{\partial z} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{z}}\right) \Rightarrow -kz = m\ddot{z} \Rightarrow z = A\cos\left(\sqrt{k/m}\, t + \varphi\right).$$

$$\frac{\partial \mathcal{L}}{\partial \theta} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) \Rightarrow 0 = mR^2\ddot{\theta} = \frac{d}{dt}\left[mR^2\dot{\theta}\right] = \dot{L}_z \Rightarrow \dot{\theta} = \text{const}$$



A block and wedge.
Block slides on the wedge, and the wedge slides on the surface, — no friction.

3 $T_M = \frac{1}{2} M \dot{q}_2^2$

$T_m = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2)$

$x = q_2 + q_1 \cos\alpha \Rightarrow \dot{x} = \dot{q}_2 + \dot{q}_1 \cos\alpha$

$y = q_1 \sin\alpha \qquad \dot{y} = \dot{q}_1 \sin\alpha$

$\Rightarrow T_m = \frac{1}{2} m \left[(\dot{q}_2 + \dot{q}_1 \cos\alpha)^2 + (\dot{q}_1 \sin\alpha)^2 \right]$

$= \frac{1}{2} m \left[\dot{q}_1^2 + \dot{q}_2^2 + 2\dot{q}_1\dot{q}_2 \cos\alpha \right]$

$U = -mgq_1 \sin\alpha \Rightarrow L = T - U = \frac{1}{2}(M+m)\dot{q}_2^2 + \frac{m}{2}(\dot{q}_1^2 + 2\dot{q}_1\dot{q}_2\cos\alpha)$
$+ mgq_1 \sin\alpha$

$\frac{\partial L}{\partial q_2} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_2}\right) \Rightarrow 0 = (M+m)\ddot{q}_2 + m\ddot{q}_1 \cos\alpha$ ①

$\Rightarrow M\dot{q}_2 + m(\dot{q}_2 + \dot{q}_1\cos\alpha) = \text{const}$ — momentum conservation

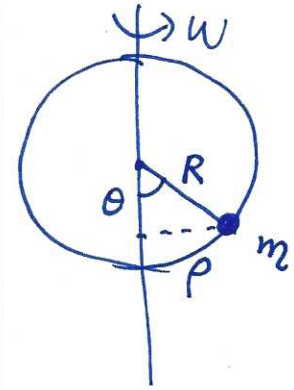
$\frac{\partial L}{\partial q_1} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_1}\right) \Rightarrow mg\sin\alpha = \frac{d}{dt}\left[m\dot{q}_1 + m\dot{q}_2\cos\alpha\right] = m(\ddot{q}_1 + \ddot{q}_2\cos\alpha)$ ②

from ① $\Rightarrow \ddot{q}_2 = -\frac{m}{M+m}\ddot{q}_1\cos\alpha \rightarrow$ plug in ②

$\Rightarrow \ddot{q}_1 = \dfrac{g\sin\alpha}{1 - \dfrac{m\cos^2\alpha}{M+m}}$ $\Rightarrow$ the time the block spends to reach the bottom is

$\Delta t = \sqrt{\frac{2l}{\ddot{q}_1}} = \sqrt{\frac{2l}{g\sin\alpha}} \cdot \sqrt{1 - \frac{m\cos^2\alpha}{M+m}}$.

4 A bead of mass m. hoop of radius R.



The hoop is rotate at ω.

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\left((R\dot{\theta})^2 + (\omega\rho)^2 \right] \leftarrow \rho = R\sin\theta$$

$$= \frac{1}{2}mR^2\left(\dot{\theta}^2 + \omega^2\sin^2\theta \right)$$

$$\mathcal{U} = mgR(1-\cos\theta)$$

$$\Rightarrow \quad \mathcal{L} = \frac{1}{2}mR^2\left[\dot{\theta}^2 + \omega^2\sin^2\theta\right] - mgR(1-\cos\theta)$$

$$\frac{\partial \mathcal{L}}{\partial\theta} = \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\right) \quad \Rightarrow \quad mR^2\omega^2\sin\theta\cos\theta - mgR\sin\theta = mR^2\ddot{\theta}$$

$$\Rightarrow \quad \ddot{\theta} = \left[\omega^2\cos\theta - g/R\right]\sin\theta$$

Equlibrium position: $\left[\omega^2\cos\theta - g/R\right]\sin\theta = 0 \Rightarrow$ ① $\theta = 0$, or, π

② $\pm\theta = \cos^{-1}\frac{g}{\omega^2 R}$

Discussion ① if $\omega^2 < g/R$, (slow rotation), there are only two equlibrium positions $\theta = 0$, or, π. Around $\theta \simeq 0$, we have

$$\ddot{\theta} = -\left[g/R - \omega^2\right]\theta \quad \Rightarrow \quad \Omega = \sqrt{g/R - \omega^2}, \text{ stable}$$

Around $\theta \approx \pi \Rightarrow \ddot{\theta} = \left(\omega^2 + g/R\right)(\theta - \pi) \rightarrow$ unstable

② if $\omega^2 > g/R \Rightarrow$ both the $\theta = 0$, and π become unstable equlibrium positions.

Check the position $\theta_0 = \cos^{-1}\frac{g}{\omega^2 R}$, and define $\theta = \theta_0 + \Delta\theta$

$\cancel{\frac{d}{d\theta}\left[\left(\omega^2\cos\theta - g/R\right)\sin\theta\right] = \omega^2}$

$$\cos\theta = \cos\theta_0 \cos\Delta\theta - \sin\theta_0 \sin\Delta\theta$$
$$\simeq \cos\theta_0 - \sin\theta_0 \,\Delta\theta$$
$$\sin\theta = \sin\theta_0 + \cos\theta_0\,\Delta\theta$$

$$\Rightarrow \omega^2\cos\theta - g/R \approx \omega^2\cos\theta_0 - g/R - \omega^2\sin\theta_0\,\Delta\theta \approx -\omega^2\sin\theta_0\,\Delta\theta$$

$\Rightarrow \ddot{\theta} \approx -\omega^2 \sin^2\theta_0\,\Delta\theta \Rightarrow$ the vibration frequency

$$\Omega = \omega\sin\theta_0 = \omega\sqrt{1-\cos^2\theta_0}$$
$$= \omega\left[1 - \left(\frac{g}{\omega^2 R}\right)^2\right]^{1/2}$$