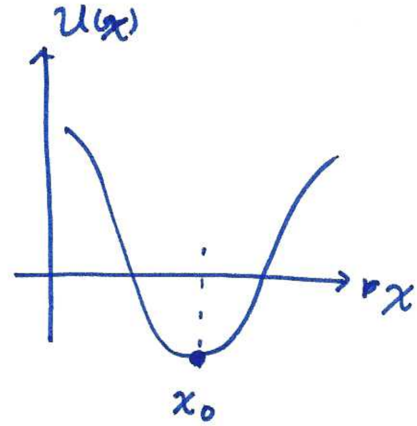
Lect 11 Oscillations (I)



For a general potential $U(x)$, its local minimal are positions of local stable equilibrium. Say, around x_0, we can expand

$$U(x) = U(x_0) + \frac{1}{2} \frac{\partial^2 U}{\partial x^2}\Big|_{x=x_0} (x-x_0)^2 + \cdots$$

This is called harmonic approximation. Shift origin to $x = x_0$, and denote $K = \frac{\partial^2 U}{\partial x^2}\Big|_{x=x_0}$. we have $U(x) = \frac{1}{2} k x^2$.

Newton's 2nd law. $F = -kx = m\ddot{x}$. Define $\omega = \sqrt{k/m}$,

$$\Rightarrow \quad \ddot{x} = -\omega^2 x.$$

Constant coefficient ODE: try solution $x \propto e^{\lambda t}$, then ODE (ordinary differential equation) $\longrightarrow$ algebra equation.

$$\Rightarrow \quad \lambda^2 e^{\lambda t} = -\omega^2 e^{\lambda t}, \Rightarrow \lambda = \pm i\omega$$

$\uparrow$ characteristic equation

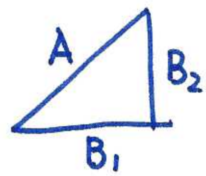
$$\Rightarrow \quad x = C_1 e^{i\omega t} + C_2 e^{-i\omega t}$$ $\longleftarrow$ superposition principle

For physical reasons. x is real number. $\Rightarrow C_1 = C_2^*$

Set $C_1 = \frac{B_1 - iB_2}{2}$ $\Rightarrow$ $x = B_1 \cos\omega t + B_2 \sin\omega t$

$$x = A\cos(\omega t - \varphi) \quad \text{where } A = \sqrt{B_1^2 + B_2^2}\,, \quad \tan\varphi = \frac{B_2}{B_1}$$

Amplitude



- energy conservation

$$\mathcal{U} = \frac{1}{2}kx^2 = \frac{1}{2}kA^2\cos^2(\omega t - \delta)$$

$$T = \frac{1}{2}m\dot{x}^2 = \frac{1}{2}mA^2\omega^2\sin^2(\omega t - \delta) = \frac{1}{2}kA^2\sin^2(\omega t - \delta)$$

$$\Rightarrow \quad \mathcal{U} + T = \frac{1}{2}kA^2\,, \quad \text{and} \quad \bar{\mathcal{U}} = \frac{1}{T}\int_0^T dt\, \mathcal{U} = \frac{1}{4}kA^2$$

$$\bar{\mathcal{U}} = \bar{T} = \frac{1}{4}kA^2$$

§ Two-dimensional oscillators

$\vec{F} = -k\vec{r}$ — centeral force field $\Rightarrow$ angular momentum conservation $\Rightarrow$ planar motion

we set the motion plane as the xy-plane

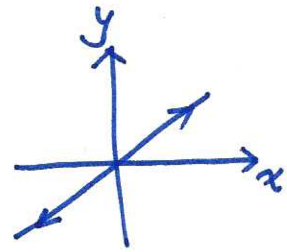
$$\left.\begin{aligned} F_x &= -kx = m\ddot{x} \\ F_y &= -ky = m\ddot{y} \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} \ddot{x} &= -\omega^2 x \\ \ddot{y} &= -\omega^2 y \end{aligned}\right.$$

* linear polarization basis

$$\left\{\begin{aligned} x(t) &= A_x\cos(\omega t - \delta_x) \\ y(t) &= A_y\cos(\omega t - \delta_y) \end{aligned}\right.$$

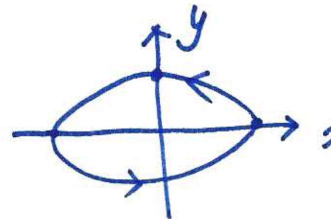
Shift the origin of time, and denote $\delta = \delta_y - \delta_x$

$$\Rightarrow \begin{cases} x(t) = A_x \cos\omega t \\ y(t) = A_y \cos(\omega t - \delta) \end{cases}$$

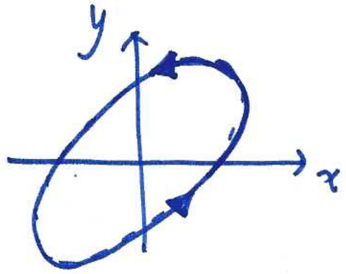


① if $\delta = 0 \Rightarrow$ the trajectory is a straight line.

② if $\delta = \pi/2 \Rightarrow \begin{cases} x(t) = A_x \cos\omega t \\ y(t) = A_y \sin\omega t \end{cases}$



③ if $\delta = \pi/4$



for general values of δ: $\quad \dfrac{y}{A_y} = \cos\omega t \cos\delta + \sin\omega t \sin\delta$

$$\Rightarrow \left(\frac{y}{A_y} - \frac{x}{A_x}\cos\delta\right) = \sin\omega t \sin\delta$$

$$\Rightarrow \left(\frac{y}{A_y} - \frac{x}{A_x}\cos\delta\right)^2 + \left(\frac{x}{A_x}\sin\delta\right)^2 = \sin^2\delta$$

$$\Rightarrow \frac{y^2}{A_y^2} + \frac{x^2}{A_x^2} - \frac{2xy}{A_x A_y}\cos\delta = \sin^2\delta$$

$$\Delta = \frac{4}{A_x^2 A_y^2}\left[\cos^2\delta - 1\right] < 0 \Rightarrow \text{ellipsis}$$

✱ Circular polarization basis

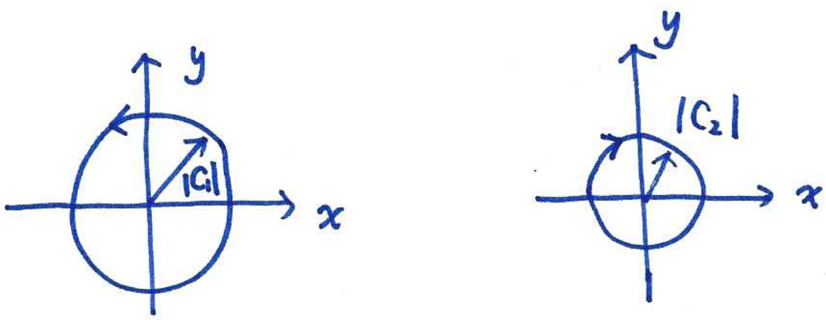
Define $z = x+iy \Rightarrow \ddot{z} = -\omega^2 z$

$\Rightarrow z = c_1 e^{i\omega t} + c_2 e^{-i\omega t} \leftarrow$ z is complex

$$c_1 e^{i\omega t} = |c_1| e^{i(\omega t - \varphi_1)} = |c_1| [\cos(\omega t - \varphi_1) + i \sin(\omega t - \varphi_1)]$$

① $c_1 e^{i\omega t}$ represent a counter-clockwise rotation with a radius $|c_1|$ and phase φ_2

② $c_2 e^{-i\omega t}$ represent a clockwise rotation.

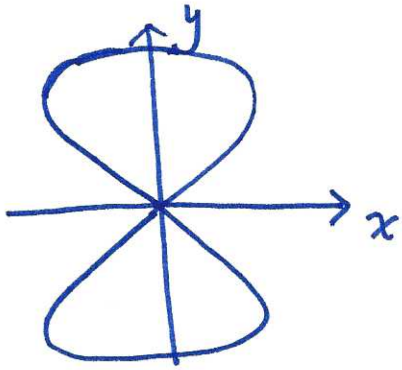


✱ oscilloscope — different frequencies along x and y

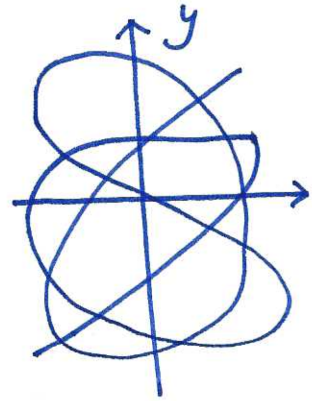
$$\begin{cases} \ddot{x} = -\omega_x^2 x \\ \ddot{y} = -\omega_y^2 y \end{cases} \rightarrow \quad \begin{aligned} x(t) &= A_x \cos \omega_x t \\ y(t) &= A_y \cos(\omega_y t - \delta) \end{aligned}$$

① if ω_x/ω_y is rational, — commensurate, the motion is still periodic. There ~~one~~ exists common period.

② if ω_x/ω_y is irrational — incommensurate. There doesn't exist a common period



$\omega_x = 2\omega_y$



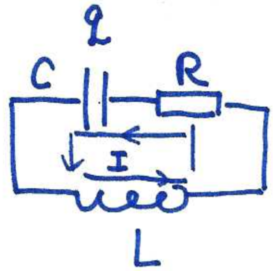
← difficult to draw.

$\omega_x = \sqrt{2}\,\omega_y$

§ Damped oscillations

① $F = -kx - b\dot{x} \Rightarrow m\ddot{x} + b\dot{x} + kx = 0.$

② LC oscillators



the emf generated by the inductance

$$\mathcal{E} = -L\frac{dI}{dt}$$

$$\mathcal{E} = IR + q/C$$

$$\Rightarrow L\frac{dI}{dt} + IR + \frac{q}{C} = 0, \quad \text{plug in } I = \dot{q}$$

$$\Rightarrow \boxed{L\ddot{q} + R\dot{q} + \frac{q}{C} = 0}$$

define $\omega_0 = \sqrt{k/m}$, $2\beta = b/m$ for oscillator

$\omega_0 = \sqrt{\frac{1}{LC}}$ $2\beta = R/L$ for LC-circuit

The damped oscillating systems can be described by the same ODES.

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0$$ — homogenous constant coefficie[nt] ODE

try $x \propto e^{\lambda t}$ $\Rightarrow$ $\lambda^2 + 2\beta\lambda + \omega_0^2 = 0$ — characteristic Eq

$$\lambda_{1,2} = -\beta \pm \sqrt{\beta^2 - \omega_0^2}$$

$$\Rightarrow x(t) = e^{-\beta t}\left[C_1 e^{\sqrt{\beta^2-\omega_0^2}\,t} + C_2 e^{-\sqrt{\beta^2-\omega_0^2}\,t}\right]$$

Discussions

① $\beta = 0$, — undamped oscillation

$$x(t) = C_1 e^{i\omega_0 t} + C_2 e^{-i\omega_0 t}$$

② $\beta < \omega_0$ — underdamped

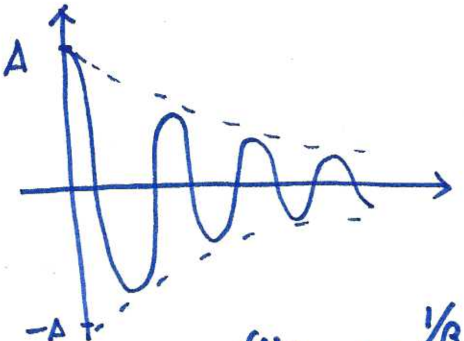
$$x(t) = e^{-\beta t}\left[C_1 e^{i\sqrt{\omega_0^2-\beta^2}\,t} + C_2 e^{-i\sqrt{\omega_0^2-\beta^2}\,t}\right]$$

$\rightarrow$ real part $x(t) = A e^{-\beta t}\cos[\omega' t - \delta]$

with $\omega' = \sqrt{\omega_0^2 - \beta^2}$

define $\dfrac{\omega_0}{2\beta} = Q$

qualitity factor



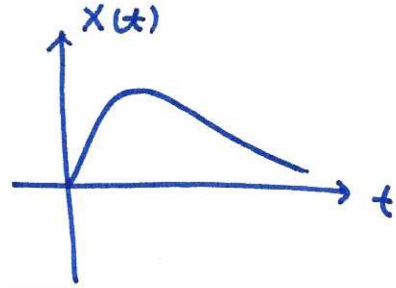
$Q \sim$ # of oscillation period in the decay time $\times \pi$.

$$Q = \frac{\omega_0}{2\beta} = \pi\frac{1/\beta}{2\pi/\omega_0} = \pi\,\frac{\text{decay ti[me]}}{\text{period}}$$

③ over damped $\beta > \omega_0$

$$x(t) = C_1 e^{-(\beta - \sqrt{\beta^2-\omega_0^2})t} + C_2 e^{-(\beta+\sqrt{\beta^2-\omega_0^2})t}$$

decay time $\sim 1/(\beta - \sqrt{\beta^2-\omega_0^2})$



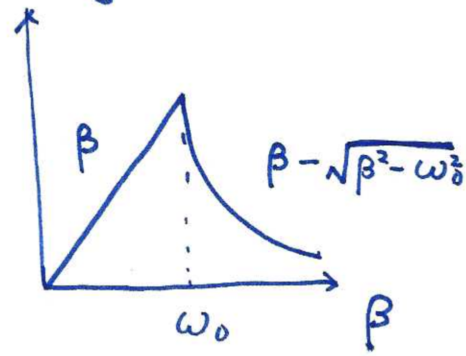
if $x(t)=0 \Rightarrow C_1 = -C_2 = C$

$$x(t) = C e^{-(\beta-\sqrt{\beta^2-\omega_0^2})t} [1 - e^{-2\sqrt{\beta^2-\omega_0^2}\,t}].$$

④ critical damping $\beta = \omega_0$, then

$$x(t) = C_1 e^{-\beta t} + C_2 t e^{-\beta t}$$

damping prameter β



$\Rightarrow$ The motion dies out most quickly at critical damping

The dissipation rate is strongest at $\beta = \omega_0$.

The disspation power $\vec{f}\cdot\vec{v} = 2m\beta v^2$

~~at~~ when increasing β, $\overline{v}^2$ also drops. $\Rightarrow$ the strongest damping is reached at intermediate level of β.