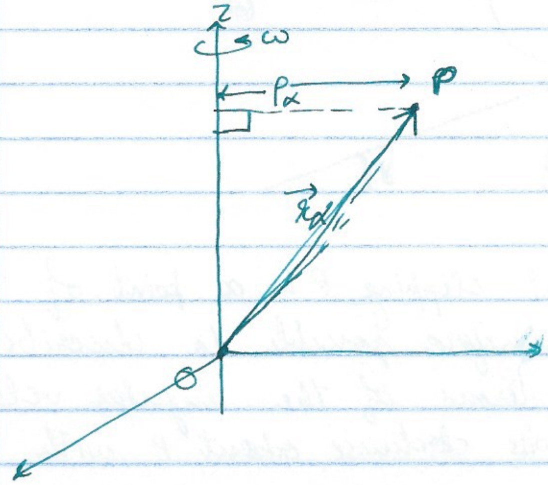
PHYS 110A HW #4

Solutions.

3.30 Taylor



① (a) $\vec{v}_\alpha = \vec{\omega} \times \vec{r}_\alpha = \omega \hat{z} \times (\rho_\alpha \hat{\rho}_\alpha + z_\alpha \hat{z}) = \omega \rho_\alpha \hat{\phi}_\alpha$

① (b) $\vec{l}_\alpha = \vec{r}_\alpha \times \vec{p}_\alpha = \vec{r}_\alpha \times (m_\alpha \vec{v}_\alpha) = m_\alpha \times (\rho_\alpha \hat{\rho}_\alpha + z_\alpha \hat{z}) \times (\omega \rho_\alpha) \hat{\phi}_\alpha$

$\vec{l}_\alpha = m_\alpha \omega \rho_\alpha^2 \hat{z} - m_\alpha \omega \rho_\alpha z_\alpha \hat{\rho}_\alpha$

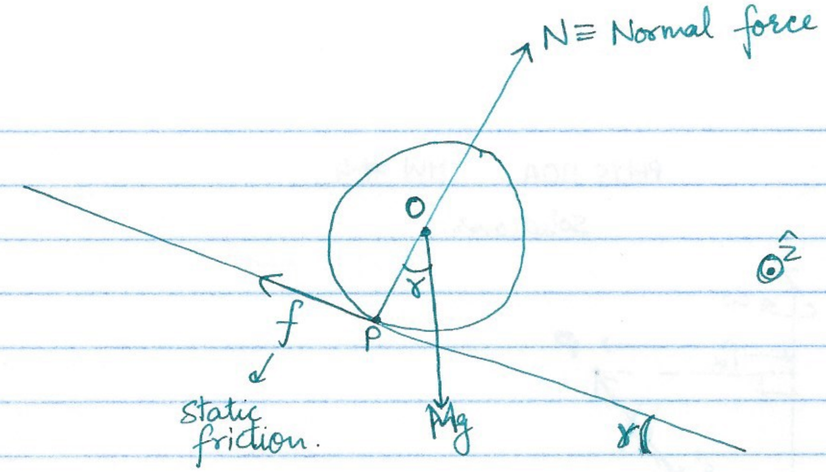
$\Rightarrow \quad (l_\alpha)_z = m_\alpha \omega \rho_\alpha^2$

① (c) $L_z = \sum_\alpha (l_\alpha)_z = \sum_\alpha m_\alpha \omega \rho_\alpha^2 = \omega \left(\sum_\alpha m_\alpha \rho_\alpha^2 \right) \equiv I\omega$

where $I = \sum_\alpha m_\alpha \rho_\alpha^2$

3.35 (a)

Taylor



Since the disk rolls without slipping, P is a point of instantaneous rest. It is therefore possible to describe the motion of the disk in terms of the angular velocity about P. Say the disk rolls clockwise about P with angular velocity ω.

Then

$$\dot{\vec{L}} = \vec{\Gamma}^{ext}$$

where $\vec{L}$ is the angular momentum about P & $\vec{\Gamma}^{ext}$ is the net external torque about P.

Now

$$\vec{L} = I\omega \quad \text{clockwise}$$

$$I = \frac{3}{2}MR^2$$

1 $$\Rightarrow \vec{L} = \frac{3}{2}MR^2\omega \text{ clockwise}$$

The forces f & N produce no torques about point P since they act at point P. The only force that produces a torque is Mg.

$$\Rightarrow \vec{\Gamma}^{ext} = \vec{PO} \times M\vec{g}$$

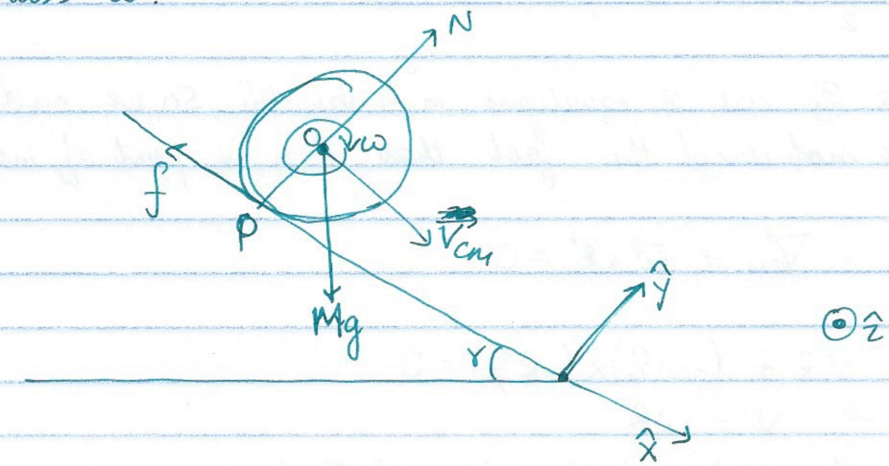
1 $$= MgR\sin\gamma \quad \text{clockwise}$$

$$\dot{\vec{L}} = \vec{\Gamma}^{ext}$$

2 $$\Rightarrow \frac{3}{2}MR^2\dot{\omega} = MgR\sin\gamma \Rightarrow R\dot{\omega} = \frac{2g}{3}\sin\gamma = \dot{v}$$

$$\Rightarrow \boxed{\dot{v} = \frac{2}{3}g\sin\gamma}$$

c Alternatively, we can analyse the motion in terms of the linear velocity of the center of mass $\vec{V}_{cm}$ & rotational velocity about the centre of mass $\vec{\omega}$.



$$\vec{V}_{cm} = V\hat{x}$$
$$\vec{\omega} = -\omega\hat{z}$$

From Newton's II law

$$\vec{F} = M\vec{a} = M\,d\vec{v}/dt$$

(1) $F_x = Mg\sin\gamma - f = M\dot{v}$ (1)

$F_y = -Mg\cos\gamma + N = 0$ (2)

Applying the torque eqn about centre of mass.

$$\dot{\vec{L}} = \vec{\Gamma}_{ext}$$

Where $\vec{\Gamma}_{ext}$ is the torque due to external forces about centre of mass & $\vec{L}$ is the angular momentum about the centre of mass.

$$\vec{L} = I\vec{\omega}$$

(1) $$= -\frac{1}{2}MR^2\omega\hat{z}$$

Now Mg does not produce a torque about CM since it acts at the CM. N does not produce a torque since its direction is parallel to the line joining the CM to the point of action of N. The only force that produces a torque is the friction.

$$\vec{\Gamma}_{ext} = \vec{OP}\times(-f\hat{x}) = -fR\hat{z}$$

Then,

(1) $$\frac{1}{2} M R^2 \dot{\omega} = f R \rightarrow (3)$$

(1), (2) & (3) are 3 equations in 4 variables. So, we need one more equ. We have not used the fact that P is a point of instantaneous rest

$$\Rightarrow \vec{V}_p = \vec{V}_{CM} + \vec{\omega} \times \vec{r} = 0$$

$$\Rightarrow V\hat{x} + (-\omega\hat{z}) \times (-R\hat{y}) = 0$$

$$\Rightarrow V = \omega R$$

Differentiating both sides wrt to time,

(1) $$\dot{V} = R\dot{\omega} \rightarrow (4)$$

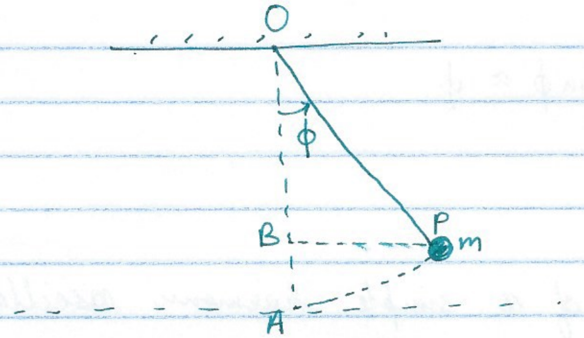
We can now solve (1), (2), (3) & (4) to get

$$\dot{V} = \frac{2}{3} g \sin\gamma$$

$$\dot{\omega} = \frac{2}{3R} g \sin\gamma$$

$$f = \frac{1}{3} M g \sin\gamma$$

4.34
(Taylor)



(a) $OA = l$

1 $OB = OP\cos\phi = l\cos\phi$

$AB = l(1-\cos\phi)$

$U(\phi) = mg \times AB$

$= mgl(1-\cos\phi)$

The pendulum's angular speed is $\dot{\phi}$. Therefore its linear speed is $l\dot{\phi}$.

1 $E = \frac{1}{2}mv^2 + U(\phi)$

$= \frac{1}{2}m(l\dot{\phi})^2 + mgl(1-\cos\phi)$

$\Rightarrow E = \frac{1}{2}ml^2\dot{\phi}^2 + mgl(1-\cos\phi)$

(b) $\frac{dE}{dt} = \frac{2}{2}ml^2\dot{\phi}\ddot{\phi} + mgl\sin\phi\,\dot{\phi}$

$\because$ the system is conservative, E is constant.

1 $\Rightarrow \frac{dE}{dt} = 0$

$\Rightarrow ml^2\ddot{\phi} = -mgl\sin\phi$

$\Rightarrow \boxed{\ddot{\phi} = -\frac{g}{l}\sin\phi}$ or $ml^2\ddot{\phi} = -mgl\sin\phi$

1

Note that ml^2 is the moment of inertia I of the mass about O & $-mgl\sin\phi$ is the torque produced by gravity on the mass about O. $\ddot{\phi}$ is the angular acceleration.

So, the equation is essentially $\Gamma = I\alpha$.

(c) For ϕ small, $\sin\phi \approx \phi$.

2 $\Rightarrow$ $\ddot{\phi} = -\frac{g}{l}\phi$

This the equation of a simple harmonic oscillator with period $T = 2\pi\sqrt{\frac{l}{g}}$.

Consider the solution

$$\phi(t) = A\sin(\omega t + \delta)$$

Note that

$$\ddot{\phi} = -\omega^2 A\sin(\omega t+\delta) = -\omega^2\phi$$

$$\Rightarrow \omega = \sqrt{\frac{g}{l}} . \qquad T = \frac{2\pi}{\omega}$$

4.38 (a) $U(\phi) = mgl(1-\cos\phi)$

$T = \frac{1}{2} ml\dot{\phi}^2$

$E = T + U$

$= \frac{1}{2} ml^2\dot{\phi}^2 + mgl(1-\cos\phi)$

$\because$ E is conserved, it is a constant of motion.

1 Then $\dot{\phi} = \sqrt{\left[(E-mgl) + mgl\cos\phi\right]\frac{2}{ml^2}}$

1 $$\frac{\tau}{4} = \int_0^{\Phi} \frac{d\phi}{\dot{\phi}}$$

$$\cancel{= \int_0^{\Phi} d\phi \sqrt{\frac{ml^2}{mgl}}}$$

$$\frac{\tau}{4} = \int_0^{\Phi} d\phi \frac{1}{\sqrt{2/ml^2\,(E - mgl + mgl\cos\phi)}}$$

At $\phi = \Phi$, $\dot{\phi} = 0$

$\Rightarrow E = mgl(1-\cos\Phi)$

$$\Rightarrow \frac{\tau}{4} = \int_0^{\Phi} d\phi \sqrt{\frac{ml^2}{2}} \frac{1}{\sqrt{mgl(\cos\phi - \cos\Phi)}}$$

1 $$\frac{\tau}{4} = \int_0^{\Phi} \left(\sqrt{\frac{l}{2g}}\right) \frac{1}{\sqrt{\cos\phi - \cos\Phi}}$$

$$\cos x = \frac{1-2\sin^2\phi}{2}$$

$$\Rightarrow \quad \tau = \int_0^{\Phi} \sqrt{\frac{l}{g}} \frac{d\phi}{\sqrt{\sin^2(\Phi/2) - \sin^2(\phi/2)}}$$

(2)

$$\tau = \frac{T}{\pi} \int_0^{\Phi} \frac{d\phi}{\sqrt{\sin^2(\Phi/2) - \sin^2(\phi/2)}}$$

where $T = 2\pi\sqrt{\frac{l}{g}}$ is the time period for small oscillations.

Substitute : $u = \frac{\sin \phi/2}{\sin \Phi/2}$ to get

$$\boxed{\tau = \frac{2T}{\pi} \int_0^1 \frac{du}{\sqrt{1-u^2}\sqrt{1-u^2A^2}}}$$