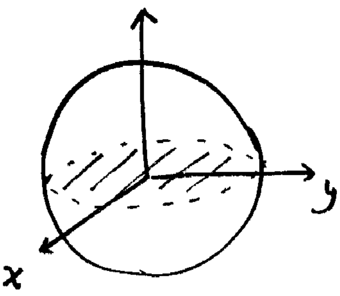
# Lect 12 Conservation laws (II)

§ conservation of momentum




example: the net force on the "northen" hemisphere of a uniformly charged solid sphere of radius R and charge Q.

Solution: method ① using the stress tensor

$$T_{ij} = \frac{1}{4\pi}(E_i E_j - \frac{1}{2}\delta_{ij} E^2) + \frac{1}{4\pi}(B_i B_j - \frac{1}{2}\delta_{ij} B^2)$$

$$f_i = \nabla_j T_{ji} - \frac{1}{c^2}\frac{\partial}{\partial t}\vec{S}$$

in our case $B=0$, $\Rightarrow \vec{S}=0$ and $T_{ij}$ only contains E-field.

$$\Rightarrow f_i = \frac{1}{4\pi}\nabla_j(E_i E_j - \frac{1}{2}\delta_{ij}E^2) \Rightarrow F_i = \frac{1}{4\pi}\oint_S (E_j E_i - \frac{1}{2}\delta_{ji}E^2)da_j$$

The boundary surface consists of two part — a hemisphere "bowl" at radius R and a circular disk of the cross section at xy plane.

① total force on the bowl

$$da = R^2 \sin\theta\, d\theta\, d\varphi\, \hat{r}, \quad E = \frac{Q}{R^2}\hat{r}$$

$$\hat{r} = \sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z}$$

The total force is along the z-axis due to the rotational sym along the z-axis

$$T_{xz} = \frac{1}{4\pi} E_x E_z = \frac{Q^2}{4\pi R^4} \sin\theta \cos\theta \cos\phi$$

$$T_{yz} = \frac{1}{4\pi} E_y E_z = \frac{Q^2}{4\pi R^4} \sin\theta \cos\theta \sin\phi$$

$$T_{zz} = \frac{1}{8\pi}\left[ E_z^2 - E_x^2 - E_y^2 \right] = \frac{1}{8\pi}\left[\frac{Q^2}{R^4}\right]\left[\cos^2\theta - \sin^2\theta\right] = \frac{Q^2}{8\pi R^4}\cos 2\theta$$

$$\Rightarrow dF_z = T_{xz}\, da_x + T_{yz}\, da_y + T_{zz}\, da_z$$

$$= \frac{Q^2}{4\pi R^4} R^2 \Big[ \underline{\sin\theta\cos\theta\cos\phi\ \sin^2\theta\cos\phi + \sin\theta\cos\theta\sin\phi\ \sin^2\theta\sin\phi}$$

$$+ \underline{\frac{1}{2}\cos 2\theta\ \sin\theta\cos\theta} \Big]\, d\theta\, d\phi$$

add together $\frac{1}{2}\sin\theta\cos\theta$

$$= \frac{\sin\theta\cos\theta}{2}\cdot\frac{Q^2}{4\pi R^2}\, d\theta\cdot d\phi$$

$$\Rightarrow F^{bowl} = \frac{Q^2}{8\pi R^2}\int_0^{\frac{\pi}{2}} \sin\theta\cos\theta\, d\theta \int_0^{2\pi} d\phi = \frac{Q^2\cdot 2\pi}{8\pi R^2}\left(\frac{-1}{4}\cos 2\theta\right)\Bigg|_0^{\frac{\pi}{2}} = \frac{Q^2}{8R^2}$$

② total force on the disk: $d\vec{a} = -r dr\, d\phi\, \hat{z}$

$$E = \frac{Q}{R^3}\vec{r} = \frac{Q}{R^3} r\left[\cos\phi\,\hat{x} + \sin\phi\,\hat{y}\right]$$

$$T_{xz} = T_{yz} = 0, \quad T_{zz} = \frac{-1}{8\pi}\left(E_x^2 + E_y^2\right) = \frac{-1}{8\pi}\frac{Q^2}{R^6} r^2$$

$$dF_z = T_{zz}\, da_z = \frac{1}{8\pi}\frac{Q^2}{R^6} r^3 dr\, d\phi \Rightarrow F_{disk} = \frac{2\pi}{8\pi}\frac{Q^2}{R^6}\frac{R^4}{4}$$

$$= \frac{Q^2}{16R^2}$$

$$\Rightarrow \text{total force } \boxed{\frac{3Q^2}{16R^2} = F^{bowl} + F^{disk}}$$

method 2: we choose the infinitly large plane $xy$, which encloses the upper hemisphere. For $r < R$, ~~cto~~ we have already got $\underline{F = \frac{Q^2}{16R^2}}$.

For $+\infty > r > R$, $\quad E = \frac{Q}{r^2}\left(\cos\phi\,\hat{x} + \sin\phi\,\hat{y}\right)$

$$T_{xz} = T_{yz} = 0, \quad T_{zz} = -\frac{(E_x^2 + E_y^2)}{8\pi} = -\frac{1}{8\pi}\frac{Q^2}{r^4}$$

$$dF_z = T_{zz}\,da_z = \frac{1}{8\pi}\frac{Q^2}{r^4}\,r\,dr\,d\phi = \frac{Q^2}{8\pi r^3}\,dr\,d\phi$$

$$\Rightarrow F_z = \frac{Q^2}{8\pi}\cdot 2\pi\int_R^\infty \frac{dr}{r^3} = \frac{Q^2}{4}\frac{1}{2R^2} = \underline{\frac{Q^2}{8R^2}}$$

add together

$$F = \frac{Q^2}{16R^2} + \frac{Q^2}{8R^2} = \frac{3Q^2}{16R^2}$$

method 3 inside bulk

$$\vec{E} = \frac{1}{r^2}\,Q\left(\frac{r}{R}\right)^3\hat{r} = \frac{r}{R^3}\,Q\,\hat{r}$$

$$d\vec{F} = \rho\vec{E}\,d\tau, \quad \rho = \frac{Q}{\frac{4\pi}{3}R^3}$$

$$dF_z = \frac{Q}{\frac{4\pi}{3}R^3}\cdot\frac{r}{R^3}Q\cos\theta\cdot\underbrace{r^2\sin\theta\,d\theta\,d\varphi}_{d\tau}$$

$$\Rightarrow F_{z,\text{upper hemi}} = \int d\vec{F}_z = \frac{Q^2}{\frac{4\pi}{3}R^6}\int_0^R r^3\,dr\int_0^{2\pi}d\varphi\int_0^{\frac{\pi}{2}}\sin\theta\cos\theta\,d\theta$$

$$= \frac{3Q^2}{4\pi R^6}\cdot\frac{R^4}{4}\cdot 2\pi\cdot\frac{1}{2} = \frac{3}{16}\frac{Q^2}{R^2}$$

## § Angular momentum of E & M field

The energy stored in the E & M field $u_{em} = \frac{1}{8\pi}E^2 + \frac{1}{8\pi}B^2$.

momentum $\vec{P}_{em} = \frac{1}{4\pi c}\vec{E}\times\vec{B}$.

we can define the angular momentum density

$$\vec{l}_{em} = \vec{r}\times\vec{P}_{em} = \frac{1}{4\pi c}\,\vec{r}\times(\vec{E}\times\vec{B}).$$

from

$$f_i = \nabla_j T_{ji} - \frac{1}{c^2}\frac{\partial}{\partial t}S_i$$

$$\tau_i = \epsilon_{ijk} r_j f_k = \epsilon_{ijk} r_j \nabla_l T_{lk} - \frac{1}{c^2}\frac{\partial}{\partial t}\epsilon_{ijk} r_j S_k$$

define $M_{ij} = T_{il} r_k \epsilon_{jlk} \Rightarrow \nabla_i M_{ij} = \nabla_i (T_{il} r_k)\epsilon_{jlk}$

$$= (\nabla_i T_{il})\, r_k \epsilon_{jlk} + \underbrace{T_{il}\delta_{ik}\epsilon_{jlk}}_{T_{kl}\epsilon_{jlk} = 0}$$

$$\Rightarrow \cancel{\nabla_j M_{ji} = \epsilon \; r_k \, \epsilon_{ilk} \nabla_{jl} T_{jl}} \quad \cancel{\epsilon_{ijk} +}$$

$$\Rightarrow \tau_i = \nabla_j M_{ji} - \frac{1}{c^2}\frac{\partial}{\partial t} l_i \qquad \text{where } l_i = \epsilon_{ijk} r_j S_k$$

$$\Rightarrow \boxed{\frac{d}{dt}\iiint d^3r\, l_i + \oiint(-M_{ji}\, da_j) = -\iiint d^3r\, \tau_i}$$

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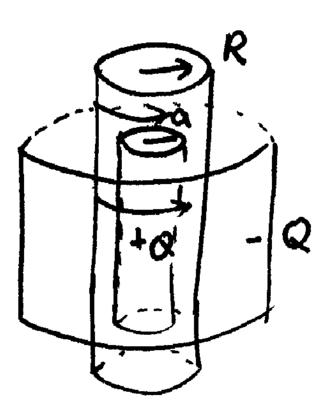


a very long solenoid with radius $R$, $n$ turns per unit length, and curren $I$. Inside and outside are two charged cylindrical shells with $\pm Q$, respectively.

whe current in the solenoid turn off gradually, we know the cylinders begin to rotate. How the angular momentum comes from?

Solution ① The B-field inside the solenoid induced by changing $I$ is

$$B \cdot l = \frac{4\pi}{c} N I \Rightarrow B = \frac{4\pi n}{c} I,$$

$$E_i \cdot 2\pi r = -\frac{1}{c}\frac{\partial}{\partial t}(B \cdot \pi R^2) = -\frac{n}{c^2} 4\pi^2 R^2 \frac{\partial I}{\partial t}$$

$$E_i = -\frac{2\pi R^2 n}{c^2 r} \frac{\partial I}{\partial t} \hat{z} \quad \text{for} \quad r > R,$$

$$E_i \, 2\pi r = -\frac{1}{c}\frac{\partial}{\partial t} B . \pi r^2 = -\frac{n}{c^2} 4\pi^2 r^2 \frac{\partial I}{\partial t}$$

$$E_i = -\frac{2\pi r}{c^2} n \frac{\partial I}{\partial t} \hat{z} \quad \text{for} \quad r < R.$$

$E_i$ is along the tangential direction $\Rightarrow$ The torque on the outter cylinder is $-b \cdot QE = -\frac{2\pi R^2 n Q}{c^2 \cdot b} \cdot b \frac{\partial I}{\partial t}$

$$\Rightarrow L_b = +\int \tau \, dt = \frac{-2\pi R^2}{c^2} n Q I \, \hat{z}$$

the torge on the inner cylinder $aQE = -\frac{2\pi a^2}{c^2} n \frac{\partial I}{\partial t}\,\hat{z}$

$$L_a = \frac{2\pi a^2}{c^2} nQI\,\hat{z}$$

$\Rightarrow$ the mechanical angular momenta together $L_a + L_b = -\frac{2\pi}{c^2} nQI(R^2-a^2$

now let us check the angular momentum density from the field.

$\vec{E} = \frac{2Q}{\ell r}\hat{e}_r$ for $a < r < b$, (by using Gauss' law).

B field only lies inside the solenoid at $r < R$, in which

$$\vec{B} = \frac{4\pi n}{c} I\,\hat{e}_z$$

$$\Rightarrow \quad \vec{p}_{em} = \frac{\vec{E}\times\vec{B}}{4\pi c} = \frac{1}{4\pi c}\cdot\frac{2Q}{\ell r}\cdot\frac{4\pi n}{c} I\,\hat{e}_r\times\hat{e}_z = \frac{2nQI}{c^2\ell r}(-\hat{e}_\theta)$$

$$\vec{l}_{em} = \vec{r}\times\vec{p}_{em} = r\,\hat{e}_r\times\frac{2nQI}{c^2\ell r}(-\hat{e}_\theta) = -\frac{2nQI}{c^2\ell}\,\hat{e}_z$$

$$\Rightarrow \quad \vec{L}_{em} = \int_a^R l_{em}\, 2\pi r\,dr = -\frac{2nQI}{c^2\ell}\pi(R^2-a^2)$$

This amount of angular momentum is transfered to mechanical angular momentum.