Lecture 11 Conservation laws (I)

§1. Gerneral statements: symmetry $\longrightarrow$ conservation laws

sym — same, metry — measure.

translation symmetry $\longrightarrow$ momentum conservation

$\mathcal{L}(\dot{x}, t)$ $\quad p = \frac{\partial \mathcal{L}}{\partial \dot{x}} \quad \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = + \frac{\partial \mathcal{L}}{\partial x} = 0 \Rightarrow \frac{d}{dt} p = 0.$

homogenuity of space

rotational symmetry $\longrightarrow$ angular momentum conservation

$\mathcal{L}(\dot{\varphi}, t)$ $\quad L = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} \quad \frac{d}{dt}(L) = \frac{\partial \mathcal{L}}{\partial \varphi} = 0 \Rightarrow \dot{L} = 0.$

Why? If the space is homogenous, one point is no more special than the other. If a particle is at rest at some place, it has no motivation to move to other place. If it's moving, it spends equal time in traveling the same length interval at any point along its trajectory.

Charge conservation is more subtle, which a result of some internal symmetry. Conservation law leads to continuity equation. Charge cannot be destroyed and cannot be created. Wherever charge changes localy, it means that there's a current flow to ~~transfer~~ charge to other place.

$$Q(t) = \int \rho(r,t)\, dr, \quad \frac{dQ}{dt} = -\oint\!\!\oint \vec{J} \cdot d\vec{a} = -\iiint \nabla \cdot \vec{J}\, dv \Rightarrow \frac{\partial \rho}{\partial t} + \nabla \cdot \vec{j} = 0$$

Continuity equation should be valid in any frame. $\Rightarrow$ it must be relativistic invariant! $(c\rho, \rho\vec{v})$, $\left(\frac{1}{c}\frac{\partial}{\partial t}, -\frac{\partial}{\partial x}\right)$ are relativstic vectors. which satisfy Lorentz transformation. $\Rightarrow$ inner product is invariant

$$\boxed{\frac{\partial}{\partial t}\rho + \nabla\cdot\vec{j} = 0}$$

§ Poynting's theorem

$$\mathcal{U} = \int d^3x \left(\frac{E^2}{8\pi} + \frac{B^2}{8\pi}\right).$$

Is this energy conserved ?
We have to take into the dissipation - the work done by E-M force.

$$\vec{F}\cdot d\vec{l} = q(\vec{E} + \frac{\vec{v}}{c}\times\vec{B})\cdot\vec{v}\,dt = q\vec{E}\cdot\vec{v}\,dt$$

$$\nabla\times\vec{B} = \frac{4\pi}{c}\vec{j} + \frac{1}{c}\frac{\partial\vec{E}}{\partial t}$$

$$\frac{dw}{dt} = \int_v \vec{E}\cdot\vec{j}\,d^3\vec{r} = \int_v d^3\vec{r}\ \vec{E}\cdot\left[\frac{c\,\nabla\times\vec{B}}{4\pi} - \frac{1}{4\pi}\frac{\partial\vec{E}}{\partial t}\right]$$

power

$$\nabla\cdot(\vec{E}\times\vec{B}) = \vec{B}\cdot(\nabla\times\vec{E}) - \vec{E}\cdot(\nabla\times\vec{B})$$

$$\Rightarrow \frac{dw}{dt} = \int d^3\vec{r}\ \frac{c\left[\vec{B}\cdot(\nabla\times\vec{E}) - \nabla\cdot(\vec{E}\times\vec{B})\right]}{4\pi} - \frac{1}{4\pi}\vec{E}\cdot\frac{\partial\vec{E}}{\partial t}$$

$$= \int d^3\vec{r}\ \frac{-1}{8\pi}\frac{\partial}{\partial t}(E^2 + B^2) - \oint \frac{c}{4\pi}(\vec{E}\times\vec{B})\cdot d\vec{a}$$

$$\Rightarrow \text{or} \int d^3\vec{r}\ \frac{\partial}{\partial t}\mathcal{U}_{em} + \oint \frac{c}{4\pi}(\vec{E}\times\vec{B})\cdot d\vec{a} = -\frac{dw}{dt}$$

The energy cost from EM to other energy.

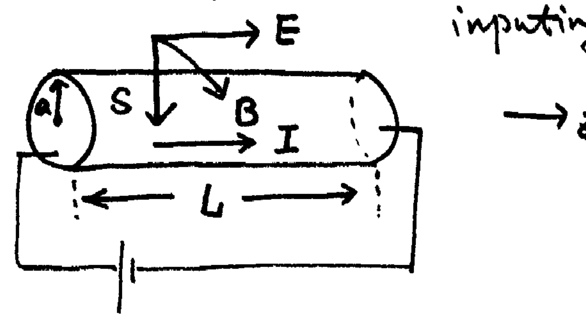
$$\frac{\partial}{\partial t} u_{em} + \frac{c}{4\pi} \nabla\cdot(\vec{E}\times\vec{B}) = -\vec{E}\cdot\vec{j} \rightarrow \text{E.M energy leaking}$$

$\Rightarrow$ EM energy flow $\vec{S} = \frac{c}{4\pi}\vec{E}\times\vec{B}$. $\quad \vec{E}\cdot\vec{j} = \frac{\partial}{\partial t} u_{mech}$

$$\boxed{\frac{\partial}{\partial t}(u_{em} + u_{mech}) + \nabla\cdot\vec{S} = 0}$$

Example: where's the energy comes from? from the field inputting into wire!

$$E = \frac{V}{L} \qquad \text{Power} = VI$$

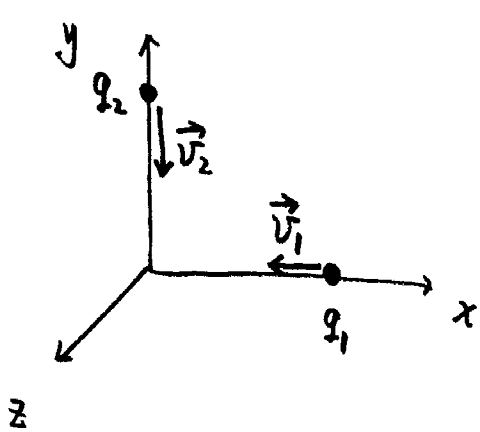


$$B = \frac{2}{rc} I$$

at the surface $\vec{S} = \frac{c}{4\pi}\vec{E}\times\vec{B} = \frac{1}{a}\,\frac{V}{2\pi L}\cdot I\ \hat{e}_z\times\hat{e}_\phi = \frac{VI}{aL\,2\pi}\hat{e}_\rho$

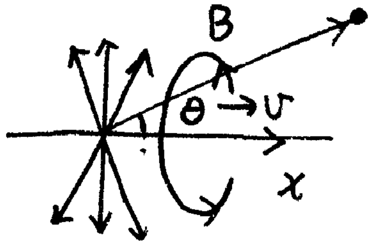
$$\Rightarrow \boxed{\oint\!\!\oint d\vec{a}\cdot\vec{S} = 2\pi a\cdot L\,\frac{VI}{aL\cdot 2\pi} = VI.}$$

§ momentum conservation:



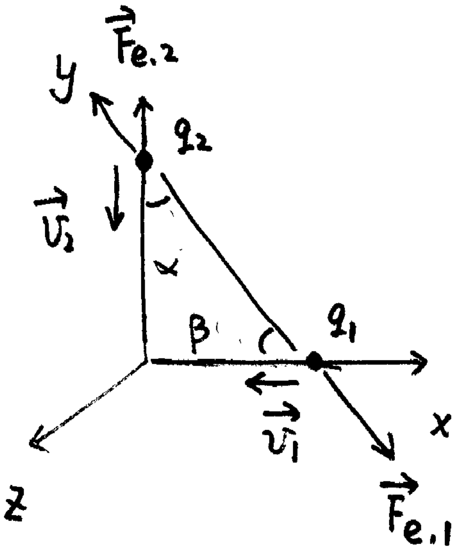
consider two charges q_1 and q_2 moving along $-\hat{x}$ and $-\hat{y}$ axis, do their interacting forces obey Newton's 3rd law?

As we learned this quarter, a moving charge generate an E-field



which is still radial, but more concentrated ~~atw~~ around the equatorial plane. It also has the magnetic field following the right hand law

$$\boxed{\vec{B} = \frac{\vec{v}\times\vec{E}}{c}, \quad \vec{E} = \frac{Q\hat{r}}{r^3}\frac{1-\beta^2}{(1-\beta^2\sin^2\theta)^{3/2}}}.$$

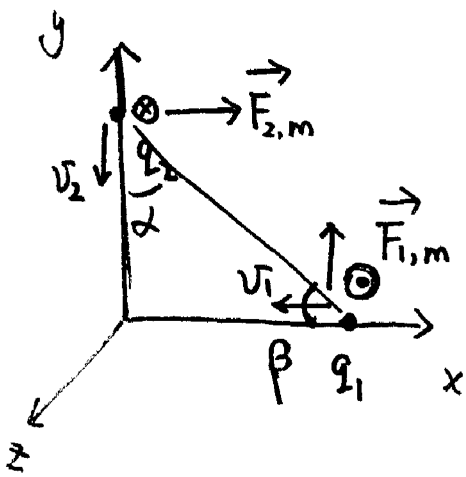


$\vec{F}_{e1}$ and $\vec{F}_{e,2}$ are opposite directions. But their magnitudes are not the same

$$F_{e,1} = \frac{q_2 q_1}{r_{12}^2}\frac{1-\beta_2^2}{(1-\beta_2^2\sin^2\alpha)^{3/2}}. \quad \beta_2 = \frac{v_2}{c}$$

$$F_{e,2} = \frac{q_2 q_1}{r_{12}^2}\frac{1-\beta_1^2}{(1-\beta_1^2\sin^2\beta)^{3/2}}. \quad \beta_1 = \frac{v_1}{c}$$

The magnetic forces are not even co-linear.



$$\vec{F}_{1,m} = \frac{v_1}{c}\frac{v_2}{c}\frac{q_1 q_2}{r^2}\frac{1-\beta_2^2}{(1-\beta_2^2\sin^2\alpha)^{3/2}}\hat{y}$$

$$\vec{F}_{2,m} = \frac{v_2}{c}\frac{v_1}{c}\frac{q_1 q_2}{r^2}\frac{1-\beta_1^2}{(1-\beta_1^2\sin^2\beta)^{3/2}}\hat{x}$$

but how to maintain momentum conservation? Field contains momentum. We cannot just consider particles. but consider they are embedded in E-M field

§ Maxwell stress tensor.

The E-M force density $\vec{F} = \int \vec{f}\, d\tau = \int \left[\rho \vec{E} + \frac{\vec{j}}{c} \times \vec{B} \right] d\tau$

$$\Rightarrow \vec{f} = \rho \vec{E} + \frac{1}{c} \vec{j} \times \vec{B} \qquad \Leftarrow \qquad \rho = \frac{1}{4\pi} \nabla \cdot E$$

$$\vec{j} = \frac{c}{4\pi} (\nabla \times \vec{B}) - \frac{1}{4\pi} \frac{\partial \vec{E}}{\partial t}$$

$$= \frac{1}{4\pi} (\nabla \cdot \vec{E}) \vec{E} + \frac{1}{4\pi} \left[(\nabla \times \vec{B}) \times \vec{B} - \frac{1}{c} \frac{\partial \vec{E}}{\partial t} \times \vec{B} \right]$$

$$-\frac{1}{c} \frac{\partial \vec{E}}{\partial t} \times \vec{B} = -\frac{1}{c} \frac{\partial}{\partial t} (\vec{E} \times \vec{B}) + \frac{1}{c} \vec{E} \times \frac{\partial}{\partial t} \vec{B} = -\frac{1}{c} \frac{\partial}{\partial t} (\vec{E} \times \vec{B}) - \vec{E} \times (\nabla \times \vec{E})$$

$$\Rightarrow \vec{f} = \frac{1}{4\pi} \left[(\nabla \cdot \vec{E}) \vec{E} - \vec{E} \times (\nabla \times \vec{E}) \right] - \frac{1}{4\pi} (\vec{B} \times (\nabla \times \vec{B})) - \frac{1}{4\pi c} \frac{\partial}{\partial t} (\vec{E} \times \vec{B})$$

$$\nabla (E \cdot E) = 2 (\vec{E} \cdot \nabla) \vec{E} + 2 \vec{E} \times (\nabla \times \vec{E}) \Rightarrow \vec{E} \times (\nabla \times \vec{E}) = \frac{\nabla E^2}{2} - (\vec{E} \cdot \nabla) \vec{E}$$

$$\Rightarrow \text{first term} = \frac{1}{4\pi} \left[(\nabla \cdot \vec{E}) \vec{E} + (\vec{E} \cdot \nabla) \vec{E} - \frac{1}{2} \nabla E^2 \right]$$

$$-\vec{B} \times (\nabla \times \vec{B}) = (\vec{B} \cdot \nabla) \vec{B} - \frac{1}{2} \nabla B^2 = \left[(\nabla \cdot \vec{B}) \vec{B} + (\vec{B} \cdot \nabla) \vec{B} - \frac{1}{2} \nabla B^2 \right]$$

$$\Rightarrow \vec{f} = \frac{1}{4\pi} \left[(\nabla \cdot \vec{E}) \vec{E} + (\vec{E} \cdot \nabla) \vec{E} + (\vec{E} \to \vec{B}) \right] - \frac{1}{8\pi} \nabla (E^2 + B^2) - \frac{1}{c^2} \frac{\partial}{\partial t} \vec{S}$$

Let's represent it in terms of $f_i = \nabla_j T_{ji} \quad \underbrace{\frac{\partial}{\partial t} S_i}_{-\frac{1}{c^2}}$

we define $T_{ij} = \frac{1}{4\pi} (E_i E_j - \frac{1}{2} \delta_{ij} E^2)$

$$+ \frac{1}{4\pi} (B_i B_j - \frac{1}{2} \delta_{ij} B^2)$$

Check

$$\nabla_i T_{ij} = \frac{1}{4\pi}\left[(\nabla\cdot E)E_j - \frac{1}{2}\nabla_j E^2\right] + (E \to B)$$
$$+ E_i \partial_i E_j$$

$\Rightarrow$

$$F_i = \int\left(\nabla_j T_{ji} - \frac{1}{c^2}\frac{\partial}{\partial t} S_i\right) d^3r$$

$$= \oiint_S T_{ji}\, da_j - \frac{1}{c^2}\frac{\partial}{\partial t}\int S_i\, d^3r$$

$$\therefore \frac{d}{dt}\Big[\underbrace{\int \frac{1}{c^2} S_i\, d^3r}_{\text{momentum density}}\Big] + \oiint \underbrace{(-T_{ji})}_{\text{momentum flow}} da_j = \underbrace{-F_i}_{\text{momentum transform to mechanical degree of freedom.}}$$

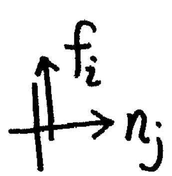
S_i : energy current

$= c^2$ mass current $\uparrow$

T_{ij} is force (per area) in the i-th direction

$\nearrow$ acting on an element of surface oriented in the j-th direction.

stress tensor.



$\Rightarrow$

$$\frac{d}{dt}\int d^3r\left(\frac{1}{c^2} S_i + f_i\right) = \iint \nabla_j T_{ji}\, d^3r$$

$$\Rightarrow \boxed{\frac{\partial}{\partial t}\left[P_{mech\,i} + P_{em\,i}\right] = \nabla_j T_{ji}}$$