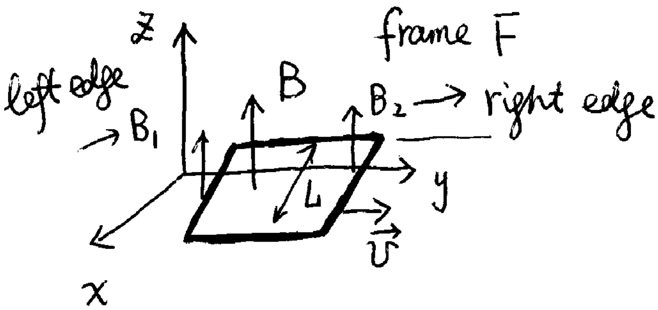
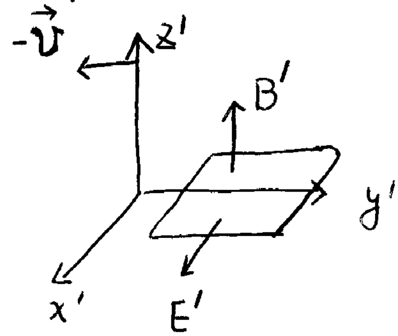
Lect 5: Electro-magnetic induction

§ a loop of wire moving in non-uniform B-field





lab frame with zero E field.

Two edges (left and right) will feel Lorentz force along the loop, and for the other edges, the Lorentz force is perpendicular to the loop.

$$\oint \vec{f}\cdot d\vec{s} = \frac{qv}{c}(B_1 - B_2)\, L \qquad \text{define emf: } \mathcal{E} = \frac{\oint \vec{f}\, d\vec{s}}{q} = \frac{Lv}{c}(B_1 - B_2)$$

or we define magnetic flux $\Phi = \iint d\vec{a}\cdot\vec{B}$

$$\frac{d\Phi}{dt} = \frac{B_2 L v\Delta t - B_1 L v \Delta t}{\Delta t} = (B_2 - B_1) L v \Rightarrow \boxed{\mathcal{E} = -\frac{1}{c}\frac{d\Phi}{dt}}$$

due to the fact that there's no magnetic charge $\nabla\cdot\vec{B} = 0$, the flux penetrating a surface only depends on the boundary. As long as the boundary is specified, it doesn't matter which surfaces you measure.

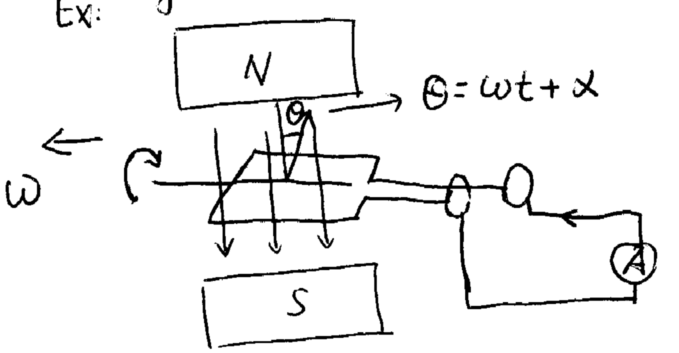
$\mathcal{E}$ can also be represented as

$$\mathcal{E} = \frac{1}{c}\oint (\vec{v}\times\vec{B})\cdot d\vec{l}$$

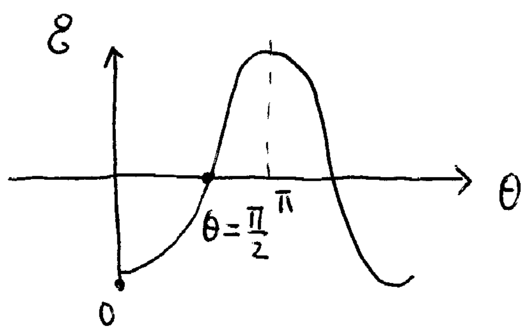
$$\mathcal{E} = -\frac{1}{c}\frac{d\Phi}{dt}$$ ← Lentz's law.

the flux generated by the induced current should be in an opposite direction to the change of Φ. (not Φ itself, but $d\Phi$!)

Ex: generator



$$\Phi = S \cdot B \quad \sin(\omega t + \alpha) \Rightarrow \mathcal{E} = -\frac{1}{c}\frac{d\Phi}{dt} = -\frac{SB\omega}{c}\cos(\omega t + \alpha)$$



§: a stationary loop in a changing B-field

In the comoving frame, there exists electric field E'. The emf is purely generated by E'. $\vec{E}' = -\frac{\vec{v}' \times \vec{B}'}{c} = \frac{\vec{v} \times \vec{B}'}{c}$

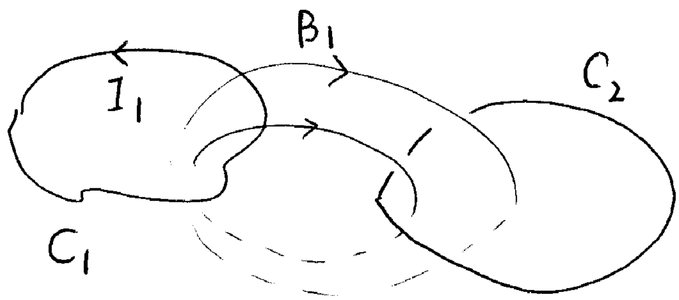
$$\oint \vec{E}' \cdot ds' = \frac{Lv}{c}(\vec{B}_1' - \vec{B}_2') \quad \text{again} \quad \mathcal{E}' = -\frac{1}{c}\frac{d\Phi'}{dt'}$$

please note that we need to use $\mathcal{E}'$, B' and t' consistently.

→ $\boxed{\text{Farady's law} \quad \oint \vec{E} \cdot d\vec{s} = -\frac{1}{c}\frac{d}{dt}\iint \vec{B} \cdot d\vec{a} \iff \nabla \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}}$

§ mutual conductance

- Consider two current loops C_1 and C_2 (position fixed,



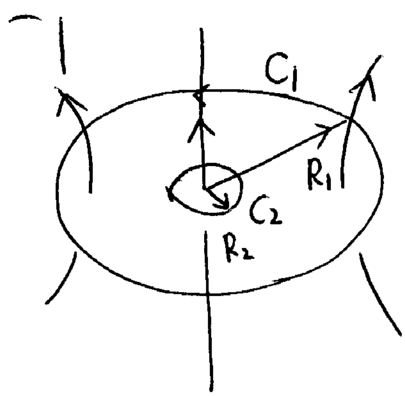
flux from I_1 through loop C_2

$$\Phi_{21} = \iint_{S_2} \vec{B} \cdot d\vec{a}_2 \overset{=}{\propto} I_1 \cdot \text{Const}$$

$$\Rightarrow \mathcal{E}_{21} = - \frac{\text{Const}}{C} \frac{dI_1}{dt} = -M_{21} \frac{dI_1}{dt}$$

Similarly we can calculate the emf in loop 1, generated by the current change of C_2.

$$\mathcal{E}_{12} = -M_{12} \frac{dI_2}{dt}$$



Ex: B_1 at the center of the ring

$$B_1 = \frac{2\pi I_1}{CR_1}, \quad \Phi_{21} = \pi R_2^2 \frac{2\pi I_1}{CR_1} = \frac{2\pi^2 I_1 R_2^2}{CR_1}$$

$$\Rightarrow \mathcal{E}_1 = -\frac{2\pi^2 R_2^2}{C^2 R_1} \frac{dI_1}{dt}, \quad \boxed{M_{21} = \frac{2\pi^2 R_2^2}{C^2 R_1}}$$

$$[M] = \text{Henry} := \frac{[\text{Volt}]}{[\text{Amp}]/[S]}$$

paramters of R_1, R_2 are not symmetric

reciprocal theorem: $\boxed{M_{21} = M_{12},}$ why?

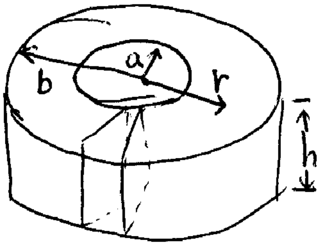
$$\left.\begin{aligned} \Phi_{21} &= \oint_{C_2} \vec{A}_{21}(\vec{r}_2) \cdot d\vec{l}_2 \\ \vec{A}_{21}(\vec{r}_2) &= \oint_{C_1} \frac{I(\vec{r}_1) d\vec{l}_1}{r_{21}} \end{aligned}\right\} \Rightarrow \Phi_{21} = \frac{I_1}{C} \oint_{C_2} \oint_{C_1} \frac{d\vec{l}_2 \cdot d\vec{l}_1}{r_{21}}$$

$$\Rightarrow M_{21} = \frac{1}{C} \oint_{C_2} \oint_{C_1} \frac{d\vec{l}_2 \cdot d\vec{l}_1}{r_{21}} = M_{12}$$

§ Self-inductance — the emf in loop C_1 generated by the change of current I_1

$$\mathcal{E}_{11} = -\frac{1}{c}\frac{d\Phi_{11}}{dt}$$

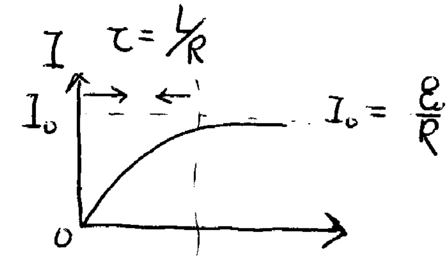
$$= -L_1 \frac{dI_1}{dt}$$



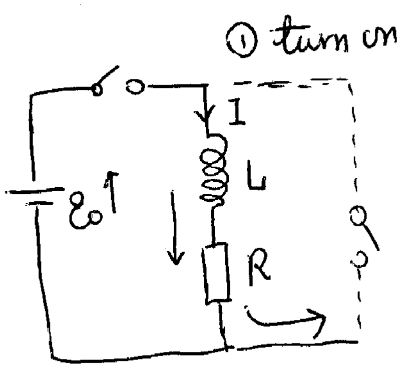
$$\Rightarrow \quad B \cdot 2\pi r = \frac{4\pi}{c} NI \Rightarrow B = \frac{2N}{cr} I$$

$$\underset{N}{\wedge}\Phi = h\int_a^b \frac{2NI}{cr}\,dr = \frac{2NIh}{c}\ln\left(\frac{b}{a}\right) \Rightarrow L = \frac{2N^2 h}{c^2}\ln\left(\frac{b}{a}\right)$$

§ Circuit contains R and L



① turn on



$$u = IR = \mathcal{E}_0 - L\frac{dI}{dt}$$

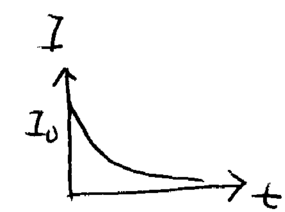
$$\text{or} \begin{cases} \dfrac{dI}{dt} = -\dfrac{R}{L}I + \dfrac{\mathcal{E}_0}{L} = -\dfrac{R}{L}\left(I - \dfrac{\mathcal{E}_0}{R}\right) \\ I(0) = 0 \end{cases}$$

$$\Rightarrow \quad I - \frac{\mathcal{E}_0}{R} = -\frac{\mathcal{E}_0}{R}e^{-\frac{R}{L}t} \Rightarrow I = \frac{\mathcal{E}_0}{R}\left(1 - e^{-\frac{R}{L}t}\right)$$

L prevent the jump of current.

② turn off

$$-L\frac{dI}{dt} = RI \Rightarrow I = I_0 e^{-R/L\,t}$$



§ Energy stored in magnetic field

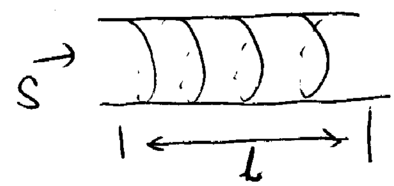
The energy dissipation on R

$$U = R\int_0^{+\infty} I^2(t)\,dt = RI_0^2 \int_0^{+\infty} e^{-\frac{2R}{L}t}\,dt = RI_0^2 \cdot \frac{L}{2R} = \frac{LI_0^2}{2}$$

This amount of energy was originally stored in the coil as magnetic

- energy $\boxed{U = \frac{1}{2} L I^2}$.

Consider a long coil



$\Phi = NSB = NSB$

$B = \frac{4\pi}{c}\frac{NI}{l}$

$\Rightarrow \Phi = \frac{4\pi}{c}\frac{N^2 S I}{l}$

$\Rightarrow L = \frac{4\pi}{c}\frac{N^2 S}{l}$

$$\Rightarrow U = \frac{1}{2}\frac{4\pi}{c}\frac{N^2 S}{l} I^2$$

$$= \left(\frac{4\pi}{c}\frac{NI}{l}\right)^2 \frac{S\cdot l}{8\pi} = \frac{B^2 \cdot Vol}{8\pi}$$

$\Rightarrow$ energy (magnetic) density $= \frac{B^2}{8\pi}$ c.f. electric energy density $\frac{E^2}{8\pi}$

- generalize to non-uniform E and B field

$$\boxed{U = \frac{1}{8\pi}\int (E^2 + B^2)\, dv}$$

Lect 6. Displacement current

what's left?

$\nabla\cdot\vec{E} = 4\pi\rho$ Gauss's law

$\nabla\cdot(\nabla\times\vec{B}) = \frac{4\pi}{c}\nabla\cdot\vec{j} \neq 0$ $\quad$ $\nabla\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t}$ Faraday's law

$\nabla\cdot\vec{B} = 0$ no magnetic monopole

$\nabla\times\vec{B} = \frac{4\pi}{c}\vec{j}$ Ampere's law

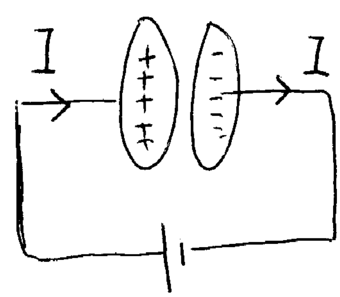
$\nabla\times\vec{B} = \frac{4\pi}{c}(\vec{j} + \vec{j}_D)$ $\quad$ $\nabla\cdot\vec{j} + \frac{\partial\rho}{\partial t} = 0$ Continuity equation

$$\nabla\cdot(\vec{j}+\vec{j}_D) = 0 \Rightarrow \nabla\cdot\vec{j}_D = +\frac{\partial\rho}{\partial t} = +\frac{1}{4\pi}\frac{\partial}{\partial t}\nabla\cdot\vec{E} \Leftarrow \text{Choose } \vec{j}_D = \frac{1}{4\pi}\frac{\partial}{\partial t}\vec{E}$$

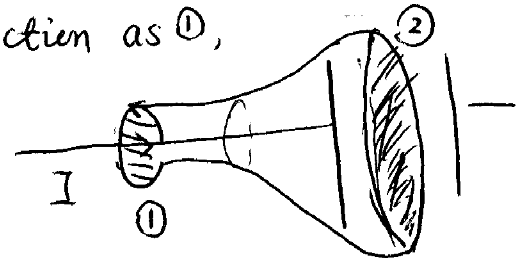
$$\Rightarrow \boxed{\nabla\times\vec{B} = \frac{4\pi}{c}\vec{j} + \frac{1}{c}\frac{\partial}{\partial t}\vec{E}}$$

Maxwell's contribution

Suppose we provide a steady charge current to a capacitor.



we know $\oint\vec{B}\cdot d\vec{l} = \frac{4\pi}{c}I$ if I choose the cross section as ①,



I can also choose the cross section as ②.

there are no I. $\oint\vec{B}\cdot d\vec{l} = 0$?. No! We can pretent that current is continous, but this time it's relatate by $\vec{j}_D = \frac{1}{4\pi}\frac{\partial\vec{E}}{\partial t}$

we can show that $E = 4\pi\sigma \Rightarrow \boxed{j_D = \frac{1}{4\pi}\frac{\partial E}{\partial t} = \frac{\partial\sigma}{\partial t} = \frac{I}{S} = j}$

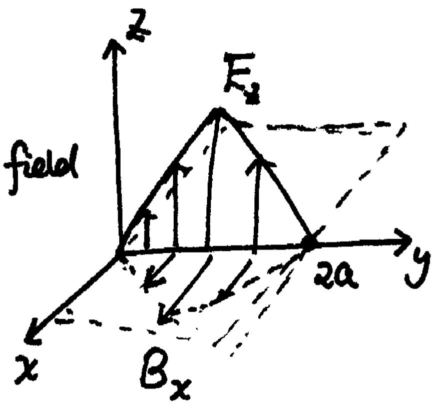
§ E & M waves

In the region without charge and current, Maxwell equations reduces to

$$\begin{cases} \nabla\times\vec{E} = -\dfrac{1}{c}\dfrac{\partial\vec{B}}{\partial t}, & \nabla\cdot\vec{E}=0 \\ \nabla\times\vec{B} = \dfrac{1}{c}\dfrac{\partial\vec{E}}{\partial t}, & \nabla\cdot\vec{B}=0 \end{cases}$$

E.M field can propogate in the free space without source. It does not need a media like sound wave. $\vec{E},\vec{B}$ do not need an underlying material field like in the elasticity theory.

Consider at $t=0$, in the region between $y=0$ and $2a$, we have the following $\vec{E},\vec{B}$ field



$(t=0)$

$$\begin{cases} E_z = E_0\dfrac{y}{a}, & (0\le y\le a) \\ E_z = E_0\left(\dfrac{2a-y}{a}\right), & (a\le y\le 2a), \end{cases}$$

$$\begin{cases} B_x = B_0\dfrac{y}{a} & (0\le y\le a) \\ B_x = B_0\left(\dfrac{2a-y}{a}\right) & (a\le y\le 2a). \end{cases}$$

they satisfy $\vec{B} = \hat{y}\times\vec{E}$

we can make it propagates by pluging in $y\to y-ct$, i.e.

$$\begin{cases} E_z = E_0\dfrac{y-ct}{a} & 0\le y-ct\le a \\ E_z = E_0\dfrac{2a-(y-ct)}{a} & a\le y-ct\le 2a \end{cases} \quad \text{and} \quad \begin{cases} B_x = B_0\dfrac{y-ct}{a} & 0\le y-ct\le a \\ B_x = B_0\left(\dfrac{2a-(y-ct)}{a}\right) & a\le y-ct \le 2a \end{cases}$$

Check in the region $0\le y-ct\le a$

$$\nabla\times E = \hat{x}\frac{\partial E_z}{\partial y} = \frac{E_0}{a}\hat{x}, \quad \nabla\times B = -\hat{z}\frac{\partial B_x}{\partial y} = -\frac{B_0}{a}\hat{z}$$

$$\nabla\cdot E = 0 \qquad\qquad \nabla\cdot B = 0$$

in the region $a \le y - ct \le 2a$

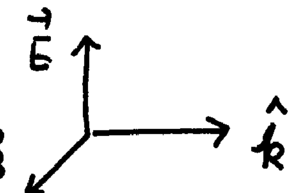
$$\nabla \times E = -\frac{E_0}{a}\hat{x}, \qquad \nabla \times \vec{B} = \frac{B_0}{a}\hat{z}$$

Similarly, in the region $0 \le y - ct \le a$ and $a \le y - ct \le 2a$;

$$\frac{\partial E}{\partial t} = -\frac{c}{a}E_0\hat{z}, \quad \frac{\partial B}{\partial t} = -\frac{c}{a}B_0\hat{x} \qquad \frac{\partial E}{\partial t} = \frac{c}{a}E_0\hat{z}, \quad \frac{\partial B}{\partial t} = \frac{c}{a}B_0\hat{x}$$

$\Rightarrow$ they satisfy $\nabla \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}$, and $\nabla \times \vec{B} = \frac{1}{c}\frac{\partial \vec{E}}{\partial t}$.

§ In general: for a $\vec{E}, \vec{B}$ configuration and propagation direction $\vec{k}$



form a triad, with $\boxed{\vec{B}(\vec{r}\cdot\hat{k} - ct) = \hat{k} \times \vec{E}(\vec{r}\cdot\hat{k} - ct)}$,

it satisfies Maxwell equation.

$$\nabla \times \vec{B} = \nabla \times (\hat{k} \times \vec{E}(\vec{r}\cdot\hat{k} - ct)) = \hat{k}(\nabla\cdot\vec{E}) - (\hat{k}\cdot\vec{\nabla})\vec{E}$$

$$= -(\hat{k}\cdot\vec{\nabla})\vec{E} = -\frac{1}{c}\frac{\partial}{\partial t}\vec{E}$$

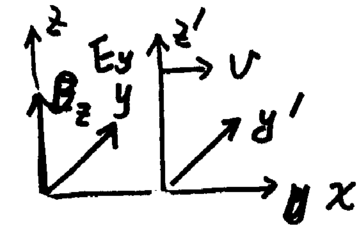
$$\nabla \times \vec{E} = \nabla \times (-\hat{k} \times \vec{B}(\vec{r}\cdot\hat{k} - ct)) = -\hat{k}(\nabla\cdot\vec{B}) + (\hat{k}\cdot\vec{\nabla})\vec{B} = \frac{1}{c}\frac{\partial}{\partial t}\vec{B}$$

From Lorentz transformation of $\vec{E}, \vec{B}$ field, we can check

$E^2 - B^2$ and $\vec{E}\cdot\vec{B}$ are invariant.

$\Rightarrow$ $|E| = |B|$ and $\vec{E} \perp \vec{B}$ cannot change.

light velocity is always "C" in any frame.



suppose in the frame F, the $\vec{E}, \vec{B}$ fields satisfy

$$\begin{cases} \vec{B} = f(x - ct) & \hat{z} \\ \vec{E} = f(x - ct) & \hat{y} \end{cases}$$

In the frame F', in which E', B' transforms

$$B_z'(x', t') = \gamma(B_z(x,t) - \beta E_y(x,t)) = \gamma(1-\beta)\ f(x-ct)$$

$$E_y'(x', t') = \gamma(E_y(x,t) - \beta B_z(x,t)) = \gamma(1-\beta)\ f(x-ct)$$

$$\begin{aligned} x &= \gamma x' - \gamma\beta ct' \\ ct &= -\gamma\beta x' + \gamma ct' \end{aligned} \Rightarrow x - ct = \gamma(1+\beta)\ (x' - ct')$$

$$\Rightarrow \begin{cases} B_z'(x', t') = \gamma(1-\beta)\ f(\gamma(1+\beta)\ (x' - ct')) = \gamma(1-\beta)\ f'(x'-ct') \\ E_y'(x', t') = \gamma(1-\beta)\ f(\gamma(1+\beta)\ (x' - ct')) = \gamma(1-\beta)\ f'(x'-ct') \end{cases}$$

$\Rightarrow B_z', E_y'$ satisfy the Maxwell equation

$$\nabla' \times \vec{B}' = \frac{1}{c}\frac{\partial \vec{E}'}{\partial t'}, \quad \nabla' \times \vec{E}' = -\frac{1}{c}\frac{\partial \vec{B}'}{\partial t'}$$